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Numerical Simulation (V4E2)
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Problem Sheet 1

1. Fundamental Solution

The function

$$G(x) = \begin{cases} \frac{1}{2\pi} \log |x| & d = 2 \\ \frac{1}{d(d-2)|B_1(0)|} |x|^{2-d} & d \geq 3 \end{cases} \tag{1}$$

is called the fundamental solution of the Laplace operator, where

$$B_\delta(x) = \{x \in \mathbb{R}^d \,|\, |x| < \delta\}. \tag{2}$$

The convolution of two functions f, g is defined by

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x - y) \, dy. \tag{3}$$

Let $f \in C_c^2(\mathbb{R}^d)$. Prove that $u := G * f$ solves $\Delta u = f$ in $\mathbb{R}^d$ for $d \geq 2$.

Hints :
- $\Delta(G * f) = G * (\Delta f) \in C^0(\mathbb{R}^d)$
- Split the integral in $B_\varepsilon(0)$ and $\mathbb{R}^d \backslash B_\varepsilon(0)$. Let $\varepsilon \to 0$ at the very end.
- $f \in W^{2,\infty}(\mathbb{R}^d)$ and $f \in W^{1,\infty}(\mathbb{R}^d)$
- $\Delta G(x) = 0$ for $|x| > \varepsilon$
- $\varepsilon^{d-1} d |B_1(0)| = |\partial B_\varepsilon(0)|$

See next page !

2. Tensor products of seperable Hilbert spaces

Let H_1, H_2 be two seperable Hilbert spaces. For $\varphi_1 \in H_1$, $\varphi_2 \in H_2$, we denote by $\varphi_1 \otimes \varphi_2$ the conjungate bilinear form on $H_1 \times H_2$ defined by

$$(\varphi_1 \otimes \varphi_2)(\psi_1, \psi_2) := \langle \psi_1, \varphi_1 \rangle_{H_1} \langle \psi_2, \varphi_2 \rangle_{H_2} \quad \forall \psi_i \in H_i,\ i = 1, 2. \tag{4}$$

Let $\mathcal{E}$ denote the space of all finite linear combinations of such bilinear forms ; on $\mathcal{E}$, we define an inner product by

$$\langle \varphi \otimes \psi, \eta \otimes \mu \rangle := \langle \varphi, \eta \rangle_{H_1} \langle \psi, \mu \rangle_{H_2} \tag{5}$$

a) Prove that $\langle \cdot, \cdot \rangle$ from (5) is well-defined and postive definite.

Therfore, $\mathcal{E}$ is a pre-Hilbert space with inner product $\langle \cdot, \cdot \rangle$. We define the tensor product $H_1 \otimes H_2$ of H_1 and H_2 as the completion of $\mathcal{E}$ under $\langle \cdot, \cdot \rangle$. Let $(\varphi_k)_k$ and $(\psi_l)_l$ be orthonormal bases of H_1 and H_2, respectively.

b) Prove that $(\varphi_k \otimes \psi_l)_{k,l}$ is an orthonormal basis of $H_1 \otimes H_2$.

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