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Numerical Simulation (V4E2)
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Problem Sheet 2

1. **On regularity of the solution of the second moment equation**
Let $D = [0,1]$ and $C_f = \mathbb{E}[f \otimes f]$ be a stationary and even function, i.e. $C_f(x,y)$, $x, y \in D$ can be written as $C_f(x-y)$ with $C_f(z) = C_f(-z)$. We look at the problem

$$\begin{cases} (\partial_x^2\partial_y^2 - \partial_x^2 - \partial_y^2 + \mathrm{Id})C_u = C_f & \text{on } D \times D, \\ \partial_y C_u = 0 & \text{on } D \times \partial D, \\ \partial_x C_u = 0 & \text{on } \partial D \times D. \end{cases} \tag{1}$$

Prove Lemma 2.3 and Lemma 2.4 from the lecture, i.e. prove first that

$$V(z) = \mathcal{J}_{C_f}(z) + A\cosh(z) + Bz\sinh(z) \tag{2}$$

solves

$$V^{(4)}(z) - 2V''(z) + V(z) = C_f(z) \qquad z \in (-1,1) \tag{3}$$

where

$$\mathcal{J}_{C_f}(z) = \int_0^z ((z-t)\cosh(z-t) - \sinh(z-t))\frac{C_f(t)}{2}\,dt \tag{4}$$

and A, B two constants. Afterwards show that for

$$A = \frac{2\mathcal{J}_{C_f}(1)}{\sinh(1)} \tag{5}$$

$$B = \frac{-\mathcal{J}_{C_f}(1)}{\sinh(1)} \tag{6}$$

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and

$$C_u^\Gamma(x,y) = -\frac{1}{\sinh(1)}\left(V'(1-x)\cosh(y) + V'(x)\cosh(1-y) + V'(1-y)\cosh(x) + V'(y)\cosh(1-x)\right) \quad (7)$$

the function

$$C_u(x,y) = V(x-y) + C_u^\Gamma(x,y) \quad (8)$$

solves (1).

2. Convergence rate for the full tensor product Galerkin discretization of a k-th moment equation

Let V be a Hilbertspace, A a continuous, V-elliptic operator and $f \in V'$. Consider the following variational problem :

$$\text{Find } u \in V : \quad \langle Au, v\rangle = \langle f, v\rangle \text{ for all } v \in V. \quad (9)$$

We use a standard Galerkin discretization based on a finite-dimensional, nested and conforming family $(V_\ell)_\ell$ with $N_\ell = \dim V_\ell$ and $V_\ell \subset V_{\ell+1} \subset V$ for all ℓ. Then we can formulate for every ℓ a discrete problem

$$\text{Find } u_\ell \in V_\ell : \quad \langle Au_\ell, v_\ell\rangle = \langle f, v_\ell\rangle \text{ for all } v_\ell \in V_\ell. \quad (10)$$

Due to the Lemma of Lax-Milgram and the Lemma of Céa we have

$$\|u - u_\ell\|_V \le C(A) \min_{v_\ell \in V_\ell} \|u - v_\ell\|_V. \quad (11)$$

We assume that for functions u in the smoothness spaces X_s with $s \in [0, s_0]$ we have the asymptotic approximation rate

$$\inf_{v_\ell \in V_\ell} \|u - v_\ell\|_V \le C\Phi(N_\ell, s)\|u\|_{X_s} \quad (12)$$

where typically for the standard Finite Element spaces $\Phi(N_\ell, s) = N_\ell^{-\frac{s}{d}}$ (d is the space dimension).

For $f \in L^k(\Omega, \mathbb{P}; V')$ the k-th moment $Z = \mathcal{M}^{(k)}u$ is the unique solution of

$$\text{Find } Z \in V^{(k)} : \quad \langle A^{(k)}Z, W\rangle = \langle \mathcal{M}^{(k)}f, W\rangle \text{ for all } W \in V^{(k)}. \quad (13)$$

Setting $\bar{V}_L := V_L \otimes \ldots \otimes V_L$ we have again a finite-dimensional, nested and conforming family $(\bar{V}_L)_L$ and can state the discrete problems

$$\text{Find } Z_L \in \bar{V}_L : \quad \langle A^{(k)}Z_L, W_L\rangle = \langle \mathcal{M}^{(k)}f, W_L\rangle \text{ for all } W_L \in \bar{V}_L. \quad (14)$$

We define for $X_0 = V$

$$[X_s]_{\text{iso}}^{(k)} := (X_s \otimes X_0 \otimes \ldots \otimes X_0) \cap (X_0 \otimes X_s \otimes X_0 \otimes \ldots \otimes X_0) \cap \ldots \cap (X_0 \otimes \ldots \otimes X_0 \otimes X_s). \quad (15)$$

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Prove that (12) **implies the following estimate :**

$$\|Z - Z_L\|_{V^{(k)}} \leq C(A,k)\Phi(N_L,s)\|Z\|_{[X_s]_{\text{iso}}^{(k)}} \tag{16}$$

Hints :

– Prove (16) only for $k = 2$
– $\|Z\|_{[X_s]_{\text{iso}}^{(k)}} = \|Z\|_{X_s \otimes X_0 \otimes \ldots \otimes X_0} + \ldots + \|Z\|_{X_0 \otimes \ldots \otimes X_0 \otimes X_s}$
– Adopt the proof of Theorem 3.2 from the lecture. Choose

$$w_L = \sum_{0 \leq i,j \leq L} (Q_i \otimes Q_j) w. \tag{17}$$

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Website : `http://chernov.ins.uni-bonn.de/teaching/ss12/StochPDEs/`