Numerical Simulation (V4E2)
Summer semester 2012
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Problem Sheet 4

1. DtN map for harmonic functions on a spheroid

Let

$$D = D(a,b) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3, \frac{x_1^2 + x_2^2}{b^2} + \frac{x_3^2}{a^2} = 1,\ a > b > 0 \right\} \tag{1}$$

be a spheroid in $\mathbb{R}^3$. In spheroidal coordinates

$$\begin{cases} x_1 &= c_0 \sinh \mu \sin \theta \cos \phi \\ x_2 &= c_0 \sinh \mu \sin \theta \sin \phi \\ x_3 &= c_0 \cosh \mu \cos \theta \end{cases} \tag{2}$$

we have

$$D = D(\mu_0, c_0) = \{(\mu_0, \theta, \varphi), \theta \in [0, \pi], \varphi \in [0, 2\pi)\}, \tag{3}$$

where $c_0 = \sqrt{a^2 - b^2} > 0$ and $a = c_0 \cosh \mu_0$, $b = c_0 \sinh \mu_0$. For a fixed spheroid D we can define a smooth one-to-one map F from the unit sphere $\mathbb{S}$ to D through

$$F(\theta, \varphi) = (\mu_0, \theta, \varphi). \tag{4}$$

Let U be a harmonic function in

$$\{(\mu, \theta, \varphi), \mu > \mu_0, \theta \in [0, \pi], \varphi \in [0, 2\pi)\}. \tag{5}$$

Then we can write

$$U(\mu, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Q_\ell^m(\cosh \mu)}{Q_\ell^m(\cosh \mu_0)} \hat{u}_{\ell,m} Y_{\ell,m}(\theta, \varphi), \tag{6}$$

where

$$Y_{\ell,m}(\theta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}P_\ell^m(\cos\theta)e^{im\varphi}, \tag{7}$$

P_ℓ^m the Legendre functions of the 1st kind and Q_ℓ^m the Legendre functions of the 2nd kind.

Prove that the normal derivative of U on D is

$$\frac{\partial U}{\partial n}(\mu_0,\theta,\varphi) = -\frac{1}{c_0\sqrt{\cosh^2\mu_0-\cos^2\theta}}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{\frac{d}{d\mu}Q_\ell^m(\cosh\mu)}{Q_\ell^m(\cosh\mu_0)}\hat{u}_{\ell,m}Y_{\ell,m}(\theta,\varphi) \tag{8}$$

By this the DtN map

$$K : U(\mu_0,\theta,\varphi) \mapsto \frac{\partial U}{\partial n}(\mu_0,\theta,\varphi) \tag{9}$$

is completely defined.

2. **Series representation and diagonal structure of the bilinear form**

Our next task is to find the Dirichlet data u for given Neuman Data g, s.t. $Ku = g$. The weak formulation reads: Find $u \in H^{\frac{1}{2}}(D)$:

$$A(u,v) := \int_D (Ku)v\,d\sigma = \int_D gv\,d\sigma \qquad \forall v \in H^{\frac{1}{2}}(D) \tag{10}$$

For $v \in L^2(D)$ it holds $v \in H^s(D)$, iff

$$\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}(1+\ell)^{2s}\hat{v}_{\ell,m}^2 < \infty, \tag{11}$$

where

$$\hat{v}_{\ell,m} = \int_{\mathbb{S}} v(F(\theta,\varphi))Y_{\ell,m}(\theta,\varphi)\,d\sigma. \tag{12}$$

Prove that

$$A(v,w) = c_0\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}H_\ell^m\hat{v}_{\ell,m}\hat{w}_{\ell,m} \tag{13}$$

with

$$H_\ell^m = \frac{-\frac{d}{d\mu}Q_\ell^m(\cosh\mu_0)}{Q_\ell^m(\cosh\mu_0)}\sinh\mu_0. \tag{14}$$

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