universitätbonn
Institut für Numerische Simulation
Rheinische Friedrich-Wilhelms-Universität Bonn
hausdorff CENTER FOR MATHEMATICS

Numerical Simulation (V4E2)
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Prof. Dr. Alexey Chernov
Claudio Bierig

Problem Sheet 6

1. On the stationary diffusion problem with random coefficients (Example 5.1)
The strong formulation of the stationary diffusion problem with random coefficients reads

$$\begin{cases} -\operatorname{div} \alpha(x,\omega)\nabla u(x,\omega) &= f(x) \quad \text{in } D \text{ for almost all } \omega \in \Omega, \\ u(x,\omega) &= 0 \quad \text{on } \partial D \text{ for almost all } \omega \in \Omega. \end{cases} \tag{1}$$

If $f \in L^2(D)$ and there exist $\alpha_-, \alpha_+ \in \mathbb{R}$ s.t. $0 < \alpha_- \le \alpha(x,\omega) \le \alpha_+ < \infty$ for all $(x,\omega) \in D \times \Omega$ there exists a unique solution and the weak formulation of (1) is:
Find $u(\cdot,\omega) \in X = H_0^1(D)$:

$$\int_D \alpha(x,\omega)\nabla u(x,\omega)\nabla v(x)\,dx = \int_D f(x)v(x)\,dx \qquad \text{for all } v \in H_0^1(D) \tag{2}$$

This can be rewritten as: Find $u \in X$:

$$\mathcal{J}(\alpha,u) = 0 \tag{3}$$

in Y', where $\mathcal{J} : Z \times X \to Y'$ is defined by

$$\langle \mathcal{J}(\alpha,u), v\rangle = \int_D \alpha\nabla u \cdot \nabla v - \int_D fv \tag{4}$$

for $Y = H_0^1(D)$, $Y' = H^{-1}(D)$, $Z = L^\infty(D)$ and $U = \{\alpha \in Z \,|\, \alpha_- \le \alpha(x) \text{ in } D\}$. Let $S : U \to X$ be the map, s.t. $\mathcal{J}(\alpha, S(\alpha)) = 0$ and define for some $\omega_0 \in \Omega$ $\alpha_0 = \alpha(\cdot,\omega_0)$ and $u_0 = S(\alpha_0)$ the solution at this point. Let W be an open neighborhood of

(α_0, u_0) in $Z \times X$. The assumptions for Theorem 5.5 and Theorem 5.6 are fullfilled for this problem. We have defined in Theorem 5.5

$$\xi_0 := \|\Gamma_0 \mathcal{J}'_\alpha(\alpha_0, u_0)\|_{Z\to X} \tag{5}$$

and η_0 being the smallest constant satisfying

$$\left.\begin{array}{l} \|\Gamma_0 \{\mathcal{J}'_\alpha(\alpha, u) - \mathcal{J}'_\alpha(\alpha_0, u_0)\}\|_{Z\to X} \\ \|\Gamma_0 \{\mathcal{J}'_u(\alpha, u) - \mathcal{J}'_u(\alpha_0, u_0)\}\|_{X\to X} \end{array}\right\} \le \eta_0 \left(\|\alpha - \alpha_0\|_Z + \|u - u_0\|_X\right) \tag{6}$$

for all $(\alpha, u) \in W$, where $\Gamma_0 := [\mathcal{J}'_u(\alpha_0, u_0)]^{-1}$. Furthermore Theorem 5.6 says, that S' is Lipschitz continuous in a neighbourhood of α_0, i.e.

$$\|S'(\alpha) - S'(\alpha_0)\|_{Z\to X} \le K \|\alpha - \alpha_0\|_Z . \tag{7}$$

for $K = 4\eta_0(1 + \xi_0)^2$.

Prove that $\xi_0 \le \frac{|u_0|_{H^1}}{\alpha_-}$ and $K \le \frac{4}{\alpha_-}\left(1 + \frac{|u_0|_{H^1}}{\alpha_-}\right)^2$.

2. **A semilinear elliptic PDE with random right hand side**

For $D \subset \mathbb{R}^d$ with $d \le 3$ and $f \in L^\infty(\Omega, L^2(D))$ we can formulate the following problem

$$\begin{cases} -\Delta u(x,\omega) + u(x,\omega)^3 = f(x,\omega) & \text{in } D \text{ for almost all } \omega \in \Omega, \\ u(x,\omega) = 0 & \text{on } \partial D \text{ for almost all } \omega \in \Omega. \end{cases} \tag{8}$$

Define the spaces X,Y and Z. Reformulate the problem in Y' as

$$\mathcal{J}(f, u) = 0. \tag{9}$$

Calculate the Frechet derivatives $\mathcal{J}'_u(f_0, u_0)$ and $\mathcal{J}'_f(f_0, u_0)$ for some $f_0 = f(\cdot, \omega_0) \in Z$ and $u_0 = S(f_0) \in X$.

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Website: `http://chernov.ins.uni-bonn.de/teaching/ss12/StochPDEs/`