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Numerical Simulation (V4E2)
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Problem Sheet 9

1. On the connection between a function and its correlation kernel

Let H, S be two separable Hilbert spaces. For any

$$f = \sum_{m \in \Lambda} f_m \otimes s_m \in H \otimes S \tag{1}$$

for some orthonormal basis $(s_m)_{m\in\Lambda}$ in S let C_f denote its correlation kernel with the representation

$$C_f = \sum_{m \in \Lambda} f_m \otimes f_m. \tag{2}$$

Furthermore let $C \in H \otimes H$ be a correlation kernel with representation

$$C = \sum_{m \in \Lambda} \lambda_m \phi_m \otimes \phi_m \tag{3}$$

where (λ_m, ϕ_m) are the eigenpairs of the associated operator $\mathcal{C}$.

Prove that $C_f = C$ iff there exists an orthonormal family $(X_m)_{m\in\Lambda} \subset S$ such that

$$f = \sum_{m \in \Lambda} \sqrt{\lambda_m} \phi_m \otimes X_m. \tag{4}$$

(Prove Theorem 7.9.)

2. An Example for the Karhunen-Loève expansion

Let $D = [-1, 1]$ and $a \in L^2(\Omega, L^2(D))$ with correlation kernel $C(x, y) = e^{-c|x-y|}$. Let $\mathcal{C}$ be its associated operator

$$(\mathcal{C}\phi)(x) := \int_D C(x, y)\phi(y)\, dy. \tag{5}$$

Compute the eigenpairs (λ_m, ϕ_m) of $\mathcal{C}$.

The Karhunen-Loève expansion of $a = \mathbb{E}[a] + \tilde{a}$ reads now

$$a(x, \omega) = \mathbb{E}[a](x) + \sum_{m=1}^{\infty} \sqrt{\lambda_m}\phi_m(x) X_m(\omega), \tag{6}$$

where

$$X_m(\omega) = \frac{1}{\sqrt{\lambda_m}} \int_D \tilde{a}(x, \omega)\phi_m(x)\, dx. \tag{7}$$

Hints: First prove that an Eigenpair (λ, ϕ) of $\mathcal{C}$ is a solution of the ODE

$$\begin{cases} \phi''(x) + \omega^2\phi(x) & = & 0 \quad -1 \le x \le 1, \\ c\phi(1) + \phi'(1) & = & 0 \quad , \\ c\phi(-1) - \phi'(-1) & = & 0 \quad , \end{cases} \tag{8}$$

where $\omega = \sqrt{\frac{2c - c^2\lambda}{\lambda}}$.

The well known solutions to this equations are

$$\phi(x) = a_1 \cos(\omega x) + a_2 \sin(\omega x), \tag{9}$$

where a_1 and a_2 are defined by the boundary conditions. Prove next that there exists nontrivial solutions for (8) only if

$$\begin{cases} c - \omega \tan(\omega) = 0 \\ \text{or} \\ c + \omega \tan(\omega) = 0. \end{cases} \tag{10}$$

Denote the solutions of the first equation by ω_n and of the second by ω_n'. Finally prove that the assosciated $L^2(D)$ normalized functions ϕ_n and ϕ_n' are defined through

$$\phi_n(x) = \frac{\cos(\omega_n x)}{\sqrt{1 + \frac{\sin(2\omega_n)}{2\omega_n}}}, \tag{11}$$

$$\phi_n'(x) = \frac{\sin(\omega_n' x)}{\sqrt{1 - \frac{\sin(2\omega_n')}{2\omega_n'}}}. \tag{12}$$

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Website: `http://chernov.ins.uni-bonn.de/teaching/ss12/StochPDEs/`