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Numerical Simulation (V4E2)
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Problem Sheet 10

We are looking a the stationary diffusion problem with random coefficients. Let D be some domain in $\mathbb{R}^d$ and $I = [-1,1]^M$, $\varrho = \left(\frac{1}{2}\right)^M$. Let u be the solution of

$$\begin{cases} -\operatorname{div} a(x,y)\nabla u(x,y) &= f(x) \quad x \in D,\ y \in I, \\ u(x,y) &= 0 \qquad x \in \partial D,\ y \in I, \end{cases} \tag{1}$$

in $L^2_\varrho(I) \otimes V$, where $V = H^1_0(D)$ and f is independent of y.

1. On the Legendre coefficients decay of the solution

Let $a(x,y) = 1 + \sum_{n=1}^M b_n y_n$ be independent of x, where $b_n > 0$ and

$$0 < a_{\text{min}} \leq a(y) \leq a_{\text{max}} < \infty. \tag{2}$$

We denote with $g \in V$ the solution of the following auxiliary problem

$$\begin{cases} -\Delta g(x) &= f(x) \quad x \in D, \\ g(x) &= 0 \qquad x \in \partial D. \end{cases} \tag{3}$$

Prove that

$$\|u_p\|_{L^\infty(I,V)} \leq \frac{\|g\|_V}{a_{\text{min}}} \frac{|p|!}{p!} e^{-\sum_{n=1}^M g_n p_n} \tag{4}$$

with

$$g_n = -\log\left(\frac{b_n}{2a_{\text{min}}}\right). \tag{5}$$

2. **Semi-discretizing the problem with Lagrange polynomials**
Let

$$a(x,y) = \bar{a}(x) + \sum_{n=1}^{M} \varphi_n(x) y_n. \tag{6}$$

We denote with $(y_\nu^p)_{\nu=1}^p$ the zeros of the p-th Legendrepolynomial. Let

$$\ell_\nu^p(y) = \prod_{\mu \neq \nu} \frac{y - y_\mu^p}{y_\nu^p - y_\mu^p} \tag{7}$$

be the Lagrange polynomials associated with (y_ν^p). We denote the tensorized Lagrange polynomials by

$$L_\nu^p(y) = \prod_{n=1}^{M} \ell_{\nu_n}^{p_n}(y_n) \tag{8}$$

and the span of all Lagrange polynomials for a fixed p with $\Lambda(p)$:

$$\Lambda(p) = \operatorname{span}\left\{ L_\nu^p \,|\, 1 \leq \nu_n \leq p_n \right\}. \tag{9}$$

Prove that the solution of (1) **in** $\Lambda(p) \otimes V$ **can be written as**

$$u(x,y) = \sum_{\nu} u_p(x) L_\nu^p(y) \tag{10}$$

where u_p **is the solution of**

$$\begin{cases} -\operatorname{div} a(x, y_\nu^p) \nabla u_p(x) &= f(x) \quad x \in D, \\ u_p(x, y_\nu^p) &= 0 \qquad x \in \partial D. \end{cases} \tag{11}$$

Hint: Show that

$$\int_I a(x,y) L_\nu^p(y) L_\mu^p(y) \varrho(y)\, dy = \delta_{\nu\mu} \lambda_\nu^p a(x, y_\nu^p), \tag{12}$$

where $\lambda_\nu^p = \prod_{n=1}^{M} \lambda_{\nu_n}^{p_n}$ and $\lambda_{\nu_n}^{p_n}$ are the weights of the Gauss-Legendre quadrature rule

$$\int_{-1}^{1} \varphi(y) \frac{1}{2}\, dy \approx \sum_{\nu=1}^{p} \varphi(y_\nu^p) \lambda_\nu^p \tag{13}$$

which is exact up to degree $2p-1$.

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Website: `http://chernov.ins.uni-bonn.de/teaching/ss12/StochPDEs/`