Numerical Algorithms

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Exercise Sheet 3.

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Exercise 1. (First Lemma of Strang)

Consider the variational problem

$$a(u,v) = \langle \ell, v\rangle \quad \text{for } v \in V$$

on $V \subset H^m(\Omega)$ for an elliptic bilinear form a. Furthermore, let there be a sequence of finite discretizations of this variational problem

$$a_h(u_h,v) = \langle \ell_h, v\rangle \quad \text{for } v \in S_h$$

on finite subspaces $S_h \subset V$. Additionally, assume that the bilinear forms a_h are equi-elliptic, so there is $\alpha > 0$ independent of h with

$$a_h(v,v) \geq \alpha \|v\|_m^2 \quad \text{for } v \in S_h.$$

Prove the first Lemma of Strang, which states that under the above assumptions there is a constant $c > 0$ so that

$$\begin{aligned}\|u-u_h\|_m \leq & c\left(\inf_{v_h\in S_h}\left\{\|u-v_h\|_m + \sup_{w_h\in S_h}\frac{|a(v_h,w_h)-a_h(v_h,w_h)|}{\|w_h\|_m}\right\}\right.\\ &\left. + \sup_{w_h\in S_h}\left\{\frac{|\langle \ell, w_h\rangle - \langle \ell_h, w_h\rangle|}{\|w_h\|_m}\right\}\right).\end{aligned}$$

(6 Points)

Exercise 2. (Fréchet derivative)

Let X, Y and Z be Banach spaces. Show that for Fréchet-differentiable functions $f\colon X \to Y$ and $g\colon X \to Y$ and a bounded bilinear operator $B\colon Y \times Y \to Z$ the function $B \circ (f,g)\colon U \to Z$ is Fréchet-differentiable and satisfies

$$D\left(B \circ (f,g)\right)(x)h = B(Df(x)h, g(x)) + B(f(x), Dg(x)h)$$

for all $x, h \in X$.

(6 Points)

Exercise 3. (Variational Problems)

Consider the bilinear form $a\colon H^1(0,1) \times H^1(0,1) \to \mathbb{R}$ defined by

$$a(u,v) := \int_0^1 x^2 u'v'\,dx.$$

a) Write down the associated classical differential equation.

b) Show that the problem of finding a minimum of

$$\frac{1}{2}a(u,u) - \int_0^1 u dx$$

does not have a solution in $H_0^1(0,1)$.

c) Show that a is not elliptic.

(0 Points)