Numerical Algorithms

Winter 2023/24
Prof. Dr. Carsten Burstedde
Hannes Brandt

UNIVERSITÄT BONN

Exercise Sheet 4.

Submission: **14.11.2023**

Exercise 1. (Stokes equations I)

Let $\Omega \subset \mathbb{R}^n$ be an open and bounded domain with smooth boundary $\partial\Omega$. The motion of an incompressible viscous fluid with velocity field $u\colon \Omega \to \mathbb{R}^n$ can be modeled with the PDE

$$\begin{aligned} -\Delta u + \nabla p &= f \quad \text{in } \Omega\,, && (1)\\ \operatorname{div} u &= 0 \quad \text{in } \Omega\,, && (2)\\ u &= u_0 \ \text{on } \partial\Omega\,. && (3) \end{aligned}$$

Here, $p\colon \Omega \to \mathbb{R}$ is the pressure, $f\colon \Omega \to \mathbb{R}^n$ is an external force density field and Δ is the componentwise Laplacian. For the spaces $X = H_0^1(\Omega)^n$ and $M = L_2(\Omega)$, the weak formulation of this problem can be stated as follows:
Find $(u, p) \in X \times M$ *such that*

$$\begin{aligned} a(u, v) + b(v, p) &= (f, v)_{L^2(\Omega)} + \ell(v)\\ b(u, q) &= 0 \end{aligned}$$

is satisfied for all $v \in X$, $q \in M$.
Determine the bilinear maps $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ and the linear map $\ell(\cdot)$. What compatibility condition does the PDE imply for u_0?

(6 Points)

Exercise 2. (Arnoldi Iteration I)

The Arnoldi iteration for a matrix $A \in \mathbb{R}^{n\times n}$ and an initial guess x is defined as

$v_1 = \frac{x}{\|x\|_2}$
for $k = 1, \dots, m$ **do**
  $u = Av_k$
  for $j = 1, \dots, k$ **do**
    $h_{jk} = v_j^T u$
    $u = u - h_{jk} v_j$
  end for
  $h_{k+1,k} = \|u\|_2$
  $v_{k+1} = \frac{1}{h_{k+1,k}} u$
end for

It corresponds to an orthogonal transformation $H_m = V_m^T A V_m$ of the matrix $A \in \mathbb{R}^{n\times n}$ to an upper Hessenberg matrix $H_m := (h_{ij})_{i,j=1,\dots,m} \in \mathbb{R}^{m\times m}$, where $m \leq n$. Now, let A be a skew-symmetric matrix, so $A^T = -A$. Which simplifications does this imply for the Arnoldi iteration? Write down the algorithm.

(6 Points)

Exercise 3. (Arnoldi Iteration II)

Another special case of the Arnoldi iteration is the symmetric Lanczos iteration, which corresponds to the application of the Arnoldi iteration to a symmetric matrix $A \in \mathbb{R}^{n \times n}$. Write down the algorithm of the Lanczos iteration. Prove that for arbitrary $\sigma \in \mathbb{R}$ the iteration always yields the same vectors v_k when applied to $A - \sigma I$ for identical start vector x. Here, v_k is defined as in exercise 2.

(0 Points)