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Exercise Sheet 6.

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Exercise 1. (Continuity in Sobolev Spaces)

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain. An admissible triangulation of Ω is a set $\mathcal{T} = \{T_1, T_2, \ldots, T_M\}$ of triangles with

$$\bigcup_{T \in \mathcal{T}} T = \overline{\Omega}$$

and for $T, T' \in \mathcal{T}$ with $T \neq T'$ the set $T \cap T'$ is either empty, contains a single corner or a whole shared edge.
Show that for $k \geq 1$ and an admissible triangulation $\mathcal{T}$ a function $v \colon \overline{\Omega} \to \mathbb{R}$ with

$$v|_T \in C^\infty(T) \quad \text{for all } T \in \mathcal{T}.$$

is in $H^k(\Omega)$ if and only if it is in $C^{k-1}(\overline{\Omega})$.

(6 Points)

Exercise 2. (Mass Matrix Recursion in 1D)

Consider a set of basis functions $\{\phi_i(x)\}$ that satisfies the recursion formula

$$\phi_i(x) = (a_i x + b_i)\phi_{i-1}(x) + c_i \phi_{i-2}(x)\,.$$

Show that the mass matrix entries $M_{ij} = \int g(x)\phi_i(x)\phi_j(x)\,\mathrm{d}x$ satisfy

$$M_{ij} = \frac{a_i}{a_{j+1}} M_{i-1,j+1} - \frac{a_i b_{j+1}}{a_{j+1}} M_{i-1,j} - \frac{a_i c_{j+1}}{a_{j+1}} M_{i-1,j-1} + b_i M_{i-1,j} + c_i M_{i-2,j}\,.$$

(6 Points)

Exercise 3. (Piecewise Linear Elements)

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded domain. For $a, f \in L_2(\Omega)$ and $u \in C^2(\Omega)$ consider the elliptic PDE

$$\operatorname{div}\left(a(x)\nabla u(x)\right) = f(x) \quad \text{on } \Omega$$

with an arbitrary fitting Dirichlet boundary condition. Based on an admissible triangulation $\mathcal{T}$ of Ω we can compute an approximate weak solution of the PDE in the function space

$$M_0^1 := \{v \in L_2(\Omega) \mid v|_T \in \mathcal{P}_1 \text{ for all } T \in \mathcal{T}\} \cap C^0(\Omega) \subset H^1(\Omega)$$

of piecewise linear functions. Prove, that the coefficient function a can be replaced by a piecewise constant function yielding the same weak solution.

(0 Points)