Numerical Algorithms

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Exercise Sheet 8.

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Exercise 1. (Mass Matrix Implementation)

Let $\Omega = [0,1]^2$ and $\phi_{ij}(x,y) = \psi_i(x)\psi_j(y)$, $1 \le i,j \le p$ a tensorized basis set on Ω. For some weight function $G\colon \Omega \to \mathbb{R}$, we approximate mass matrix entries $M_{ij,kl} = \int_\Omega \phi_{ij}\phi_{kl}G$ via tensorized numerical integration

$$M_{ij,kl} \approx \sum_{\alpha,\beta} w_\alpha w_\beta \phi_{ij}(x_\alpha, x_\beta)\phi_{kl}(x_\alpha, x_\beta)G(x_\alpha, x_\beta) \tag{1}$$

with weights $w_1, \dots, w_p$ and nodes $x_1, \dots, x_p$. Derive an algorithm which computes the matrix-vector multiplication $y = Mu$ in $\mathcal{O}(p^3)$.

(6 Points)

Exercise 2. (Tensor-Product Elements)

On the d-dimensional hypercube $Q_{\text{ref}} = [0,1]^d$ we consider for some $m \in \mathbb{N}$ the space

$$\mathcal{Q}_m := \text{span}\left\{(x_1, ..., x_d) \mapsto \psi_1(x_1) \cdot ... \cdot \psi_d(x_d) \,\middle|\, \quad \psi_i \text{ polynomial of degree } \le m\right\},$$

i.e. $\mathcal{Q}_m$ consists of all polynomials in the variables $x_1, ..., x_d$ such that each x_i appears only with exponents at most m.

a) Given arbitray $0 = z_0 < z_1 < ... < z_m = 1$, show that there is a nodal basis for $\mathcal{Q}_m$ with respect to the point set

$$Z = \{(z_{i_1}, ..., z_{i_d}) \,|\, i_1, ..., i_d = 0, ..., m\}.$$

Now we consider only $d = 2$ and a domain $\Omega \subset \mathbb{R}^2$ being partitioned in a *rectangular* mesh, i.e. all cells of the mesh are rectangles w.l.o.g. with edges parallel to the coordinate axes. Additionally, we denote a Finite Element by a triple (T, Π, Σ), where T is a reference polyhedron, Π is a finite dimensional subspace of $C(T)$ and Σ is a basis of Π. The nodal interpolation points on a shared edge are identified with their counterpart of the respective neighboring edge.

b) Show that the finite element (Q_{ref}, $\mathcal{Q}_m$, point evaluations at Z) generates an $H^1(\Omega)$-conforming finite element space on Ω.

c) Does the finite element (Q_{ref}, $\mathcal{Q}_3$, point evaluations at Z) generate an $H^2(\Omega)$-conforming finite element space on the rectangular mesh?

(2+2+2 Points)