Numerical Algorithms

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Exercise Sheet 9.

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Exercise 1. (Anisotropic Diffusion)

We consider the anisotropic Poisson equation

$$-\left(\alpha\frac{\partial^2}{\partial x^2}+\beta\frac{\partial^2}{\partial y^2}\right)u = f \quad \text{in } \Omega=(0,1)^2,$$
$$u = 0 \quad \text{on } \Gamma := \partial\Omega$$

with $\alpha,\beta\in\mathbb{R}$ and $f\in L^2(\Omega)$.

a) Derive the weak formulation of the anisotropic Poisson equation in $H^1(\Omega)$.

b) We choose a tensor product approach for discretization. We consider a triangulation that partitions $\overline{\Omega}$ into a regular $N\times N$ grid of squares. As a basis we use the classic 'hat functions', which are 1 on one node/corner of the grid, 0 on every other node/corner and the tensor product of piecewise linear functions on every cell of the grid. By ϕ_{ij} for $i,j=0,\dots,N$ we denote the hat function that is 1 on node (i,j) of the grid.

   Compute the non-zero entries of the mass and the stiffness matrix. So, for an arbitrary basis function ϕ_{ij} compute only the 3×3 matrix representation of non-zero mass/stiffness matrix entries

$$\begin{pmatrix}\bar{m}_{i-1,j-1} & \bar{m}_{i-1,j} & \bar{m}_{i-1,j+1}\\ \bar{m}_{i,j-1} & \bar{m}_{i,j} & \bar{m}_{i,j+1}\\ \bar{m}_{i+1,j-1} & \bar{m}_{i+1,j} & \bar{m}_{i+1,j+1}\end{pmatrix}$$

   with $\bar{m}_{k,l}=M_{kl,ij}$.

   The tensor product approach allows to reduce many computations to 1D. You can reuse results from the mass matrix computation for the stiffness matrix.

(2+6 Points)