Numerical Algorithms

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Exercise Sheet 1.

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Exercise 1. (Gauss Theorem)

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with piecewise smooth boundary $\partial\Omega$. Let $f \in \mathcal{C}^1(\overline{\Omega})$, $g \in \mathcal{C}^2(\overline{\Omega})$ and $u \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^d)$ a vector field. Based on the Gauss theorem

$$\int_\Omega \partial_i f = \int_{\partial\Omega} \nu_i f \quad \forall i = 1, \dots, d$$

with outside normal vector ν prove the following equations.

a) $\displaystyle\int_\Omega \nabla \cdot u = \int_{\partial\Omega} \nu \cdot u$

b) $\displaystyle\int_\Omega \nabla f \cdot \nabla g = -\int_\Omega f \Delta g + \int_{\partial\Omega} \nu \cdot f \nabla g$

(3+3 Points)

Exercise 2. (Lax-Milgram Theorem)

Let V be a closed, convex set in an Hilbert space H and $a\colon H \times H \to \mathbb{R}$ an elliptic (i.e. continuous and coercive) bilinear form. Prove that for every $\ell \in H'$, where H' is the dual space of H, the variational problem

$$J(v) \coloneqq \frac{1}{2} a(v, v) - \langle \ell, v \rangle \longrightarrow \min !$$

has a unique solution in V.
You can show that all minimizing sequences are Cauchy sequences using the parallelogram identity.

(8 Points)

Exercise 3. (Céa Lemma, does not count towards admission)

For a $H^m(\Omega)$-elliptic bilinear form $a(\cdot,\cdot)$ consider the solution $u \in H^m(\Omega)$ of the variational problem

$$a(u, v) = \langle \ell, v \rangle \quad \text{for } v \in H^m(\Omega).$$

where ℓ is a linear form over $H^m(\Omega)$. Let S_h be a subspace of $H^m(\Omega)$ and u_h the solution of the variational problem restricted to S_h. Prove that

$$\|u - u_h\|_m \le \frac{C}{\alpha} \inf_{v_h \in S_h} \|u - v_h\|_m$$

for constants $C > 0, \alpha > 0$.

(0 Points)