Numerical Algorithms

Winter 2023/24
Prof. Dr. Carsten Burstedde
Hannes Brandt

UNIVERSITÄT BONN

Exercise Sheet 10.

Submission: **09.01.2024**

Exercise 1. (Vandermonde Matrices)

On the reference triangle spanned by the nodes $(0,0),(0,1)$ and $(1,0)$ we consider the corresponding nodal basis of linear hat functions $\phi_{i=1,\dots,3}$. To compute their mass and stiffness matrix, one can use second order quadrature with the evaluation points

$$\xi_1 = \left(\frac{1}{6}, \frac{1}{6}\right), \quad \xi_2 = \left(\frac{4}{6}, \frac{1}{6}\right), \quad \xi_3 = \left(\frac{1}{6}, \frac{4}{6}\right)$$

and weights $\omega_1 = \omega_2 = \omega_3 = \frac{1}{6}$. Compute the Vandermonde matrix $V \in \mathbb{R}^{3\times 3}$, the derivative matrices D^l and the gradient Vandermonde matrices V^l for $l = 1, 2$ as introduced in the lecture.

(6 Points)

Exercise 2. (Programming Task III)

Using the results from Exercise Sheet 7 we want to solve the system of linear equations

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \vec{u}_1 \\ \vec{u}_2 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \vec{f}_1 \\ \vec{f}_2 \end{pmatrix} \tag{1}$$

for u, given the stiffness matrix A, the mass matrix M and the right hand side f. Here, $\vec{f}$ and $\vec{u}$ denote the evaluation of f and u on the nodes defining our nodal basis and A_{ij}, M_{ij} denote block matrices. The index 1 represents the interior nodes, while 2 represents the boundary nodes.

We assume $\vec{u}_2$ is uniquely defined by Dirichlet boundary conditions. Thus, it remains to solve for $\vec{u}_1$. Using (1) we get

$$A_{11}\vec{u}_1 = \begin{pmatrix} M_{11} & M_{12} \end{pmatrix} \vec{f} - A_{12}\vec{u}_2 =: \vec{b}_1. \tag{2}$$

This system of linear equations for $\vec{u}_1$ can now be solved using the preconditioned CG method. The lumped mass matrix

$$\overline{M}_{11} = \operatorname{diag}\left(M_{11} \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}^T \right)$$

is a common preconditioner that can be computed efficiently.

a) We continue operating on the triangle mesh specified by the files `box.4.node` and `box.4.ele` available on the Florida State university website. The last entry of a row in a `.node` file is 0 if and only if the corresponding node lies in the interior of the mesh. Based on this information create a lookup table of the boundary status of all nodes.

b) Compute the vector $\vec{b}_1$ from (2) using the mass and stiffness matrix matrix-vector product from Programming Task II and the lookup table created previously.

c) Implement the preconditioned CG method for (2). The implementation should operate matrix-free. So, both the system matrix A_{11} and the preconditioner P should be represented just by a matrix-vector product.

Implement an appropriate stopping criterion for the method. Especially, enforce a canonical upper bound on the number of iterations and compute a criterion based on the residual in each iteration. Your stopping criterion should be scaling-invariant. So, the number of iterations should not be affected by scaling up (2) by any $c \in \mathbb{R}_{>0}$ to $c\, A_{11}\vec{u_1} = c\,\vec{b_1}$.

d) For $\Omega := (0,3)^2 \setminus (1,2)^2$ and $\Gamma := \partial\Omega$ we consider

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega \\ u &= xy && \text{on } \Gamma. \end{aligned}$$

Use the preconditioned CG method to solve this system. The solution $u(x,y) = xy$ is part of the global function space spanned by our 'hat function' FEM basis, so this example is convenient for verification of the implementation. Try different preconditioners like the identity matrix, the matrix $\operatorname{diag}(a_{11}, \ldots, a_{NN})$ and the lumped mass matrix. Plot the residual errors and compare their convergence.

e) On the same domain we consider the Dirichlet boundary problem

$$\begin{aligned} -\Delta u &= 4\pi^2 \sin(2\pi x)\cos(2\pi y) && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma \cap \{x \mid x \in \{0,1,2,3\}\} \\ u &= \sin(2\pi x) && \text{on } \Gamma \cap \{y \mid y \in \{0,1,2,3\}\}. \end{aligned}$$

Again, apply the preconditioned CG method with different preconditioners. Compare your results to the analytical solution $u(x,y) = \sin(2\pi x)\cos(2\pi y)$, plot the errors and compare the convergence of the preconditioners.

(0 Points)