Numerical Algorithms

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Exercise Sheet 13.

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Exercise 1. (Searching Partitions)

For $d, L, P \in \mathbb{N}_{>0}$ let $F = (I_L)_p \in [0, 2^{dL}]^{P+1} \cap \mathbb{N}^{P+1}$ the array of first local element Morton indices, as introduced in the lecture. Propose an $\mathcal{O}(\log P)$ algorithm `find_range` (F, ℓ, I_L), that given the Morton index (ℓ, I_L) with $0 \leq \ell \leq L$ of a mesh element, computes the range of all processes that own a part of the mesh elements area.

(0 Points)

Exercise 2. (Recap I - Fréchet Derivative)

Let X, Y, Z Banach spaces, $U \subset X$ an open subset and $f : U \to Y$ Fréchet-differentiable at $x \in U$. Furthermore, let $y = f(x) \in V$ for an open subset $V \subset Y$ and $g : V \to Z$ differentiable at y. Prove that $g \circ f$ is Fréchet-differentiable at x with

$$D(g \circ f)(x) = Dg(f(x)) \circ Df(x). \tag{1}$$

(0 Points)

Exercise 3. (Recap II - Weak Formulation)

Let $\Omega \subset \mathbb{R}^3$ be open and bounded with smooth boundary and outer normal ν. Based on the curl operator

$$\operatorname{curl} u := \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix} \in \mathbb{R}^3 \tag{2}$$

we define

$$H(\Omega, \operatorname{curl}) := \left\{ u \in L_2(\Omega)^3 \mid \operatorname{curl} u \text{ exists weakly and } \operatorname{curl} u \in L_2(\Omega)^3 \right\} \tag{3}$$

with scalar product $(u, v) := (u, v)_{L_2} + (\operatorname{curl} u \cdot \operatorname{curl} v)_{L_2}$. This is a Hilbert space, with $H_0(\Omega, \operatorname{curl})$ the modification with vanishing boundary $u \times \nu = 0$. For $F \in L_2(\Omega)^3$, derive the weak formulation for the PDE

$$-\operatorname{curl}\operatorname{curl} u = F \quad \text{in } \Omega \tag{4}$$

$$u \times \nu = 0 \quad \text{on } \partial\Omega \tag{5}$$

in $H_0(\Omega, \operatorname{curl})$.

(0 Points)

Exercise 4. (Recap III - Weak Solution)

Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary, $\alpha \in \mathbb{R}$ with $\alpha > 0$, and $\kappa \in L_\infty(\Omega)$ with $\kappa \geq 0$ almost everywhere in Ω. Consider the PDE

$$\begin{aligned} -\alpha \Delta u + \kappa u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

with right hand side $f \in L_2(\Omega)$.

- Derive the variational formulation of the PDE.
- Show that there exists a unique weak solution $u \in H_0^1(\Omega)$ to the PDE and that the solution depends continuously on f.
- Given a basis $\{\varphi_j\}_{j=1}^N$ of a finite-dimensional subspace $V_h \subset H_0^1(\Omega)$. State the linear system arising from the Galerkin discretization.

(0 Points)