Numerical Algorithms

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Exercise Sheet 14.

Exercise 1. (Recap IV - CG Method)

Consider the conjugate gradient and the preconditioned conjugate gradient method as introduced in the lecture.

a) Propose two different stopping criteria for the CG method. The criteria should be scaling invariant, so scaling the entire system of linear equations by a constant $0 < \alpha \in \mathbb{R}$ should not affect the total number of iterations. How can both stopping criteria be adapted to the PCG method?

b) For the search directions $\mathbf{p}_i$ and the residuals $\mathbf{z}_i$ of the PCG method prove

$$\text{span}\{\mathbf{p}_0, \dots, \mathbf{p}_k\} = \text{span}\{\mathbf{z}_0, \dots, \mathbf{z}_k\}$$

for k less or equal the total amount of iterations.

(0 Points)

Exercise 2. (Recap V - Tensor Basis)

For one-dimensional functions $\psi_i \colon [-1, 1] \to \mathbb{R}$ we define a two-dimensional tensor basis of functions

$$\psi_{ij}(x, y) := \psi_i(x)\psi_j(y)$$

on the reference element with domain $[-1, 1]^2$. Reduce the computation of the element-local stiffness matrix A with

$$A_{ij,kl} = \int_{-1}^{1}\int_{-1}^{1} \nabla\varphi_{ij}(x, y)\nabla\varphi_{kl}(x, y)dxdy$$

to the computation of one-dimensional integrals.

(0 Points)

Exercise 3. (Recap VI - Element Stiffness Matrix)

To solve the Poisson equation on $\Omega = (0, N) \times (0, N)$ with $N \in \mathbb{N}$ we partition $\overline{\Omega}$ into a regular $N \times N$ grid of squares. As a basis we use the bilinear 'hat functions' ϕ_{ij} that are 1 on node/corner (i, j) of the grid and 0 on every other node/corner. The element-local stiffness matrix of a cell has the form

$$\begin{pmatrix} \alpha & \beta & \gamma & \beta \\ \beta & \alpha & \beta & \gamma \\ \gamma & \beta & \alpha & \beta \\ \beta & \gamma & \beta & \alpha \end{pmatrix}$$

with $\alpha + 2\beta + \gamma = 0$.

The non-zero entries of the global stiffness matrix for a basis function ϕ_{ij} can be given in the form of a 3×3 matrix

$$\begin{pmatrix} \bar{a}_{i-1,j-1} & \bar{a}_{i-1,j} & \bar{a}_{i-1,j+1} \\ \bar{a}_{i,j-1} & \bar{a}_{i,j} & \bar{a}_{i,j+1} \\ \bar{a}_{i+1,j-1} & \bar{a}_{i+1,j} & \bar{a}_{i+1,j+1} \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

with $\bar{a}_{kl} = A_{kl,ij}$, like in exercise 1b) of sheet 9.

a) What is the intuition behind the equation $\alpha + 2\beta + \gamma = 0$?

b) Deduce the values α, β and γ from the given non-zero entries of the global stiffness matrix.

c) Instead of the weak formulation of the Poisson Equation now consider the bilinear form

$$a(u,v) := \int_\Omega \nabla u^T \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \nabla v$$

with $c, d \in \mathbb{R}_+$. How can the α, β, γ representation of the element-local stiffness matrix be adapted to this change?

(0 Points)