Exercise sheet 9 To be handed in on **Thursday, 23.06.2016**

Generalized interpolation, integration

1 Group exercises

G 1. (Integration in kernel spaces)

Let Ω be some set and $\rho : \Omega \to [0, \infty)$ a density function. Further let $\mathcal{H}$ be a Hilbert space of real-valued functions on Ω with kernel k such that $\int_\Omega \sqrt{k(x,x)}\rho(x)dx < \infty$.

a) Show that $\mathcal{H}$ consists of ρ-integrable functions and that the integration functional $\mathrm{Int}(f) = \int_\Omega f(x)\rho(x)dx$ is continuous.

b) Determine the function $h \in \mathcal{H}$ that represents the integration functional Int.

Solution.

a) For all $f \in \mathcal{H}$, we have

$$\int_\Omega |f(x)|\rho(x)dx = \int_\Omega |\langle f, k(x,\cdot)\rangle|\rho(x)dx \le \|f\|_k \int_\Omega \|k(x,\cdot)\|_k \rho(x)dx < \infty.$$

Since $\|k(x,\cdot)\|_k = \sqrt{k(x,x)}$, we conclude that every $f \in \mathcal{H}$ is integrable with respect to the density ρ. Moreover,

$$|\mathrm{Int}(f)| \le \mathrm{Int}(|f|) \le \|f\|_k \int_\Omega \sqrt{k(x,x)}\rho(x)dx,$$

hence Int is a bounded linear functional, which is equivalent to being a continuous linear functional.

b) By Riesz' representer theorem, there is a function $h \in \mathcal{H}$ such that $\mathrm{Int}(f) = \langle h, f\rangle$. This function is pointwise given by

$$h(x) = \langle h, k(x,\cdot)\rangle = \int_\Omega k(x,y)\rho(y)dy.$$

□

G 2. (Representation of general bounded linear functionals)

Let k be a kernel and $\mathcal{H}$ its native Hilbert space. Let $\lambda, \mu \in \mathcal{H}^*$ be two continuous functionals. Show that $\lambda^2 k(\cdot, y) \in \mathcal{H}$ and

$$\lambda(f) = \langle f, \lambda^2 k(\cdot, y)\rangle_k \qquad \text{for all } f \in \mathcal{H}.$$

Moreover, show that

$$\langle \lambda, \mu\rangle_{\mathcal{H}^*} = \lambda^1 \mu^2 k(x,y).$$

Solution. Let us write $k_x(\cdot) = k(x, \cdot)$, that is, we consider the kernel k as a function of the second argument with parameter x. Since $\lambda \in \mathcal{H}^*$, there is by Riesz' representer theorem a function $u_\lambda \in \mathcal{H}$ such that $\langle u_\lambda, f\rangle = \lambda(f)$ for all $f \in \mathcal{H}$. Since k_x represents the evaluation at the point x, we have $\lambda(k_x) = \langle u_\lambda, k_x\rangle_k = u_\lambda(x)$. Hence, $\lambda^2 k(\cdot, y) = u_\lambda \in \mathcal{H}$. Analogously, we find a function u_μ such that $\mu(k_x) = \langle u_\mu, k_x\rangle_k = u_\mu(x)$. Then,

$$\langle \lambda, \mu\rangle_{\mathcal{H}^*} = \langle u_\lambda, u_\mu\rangle_k = \lambda(u_\mu) = \lambda(\mu(k_.)) = \lambda^1\mu^2 k(x, y).$$

□

G 3. (Differentiable kernels)

Let $\Omega = [0, 1]$ and $k(x, y)$ be a kernel on Ω^2 which has continuous partial derivatives $\partial_x k$, $\partial_y k$. Assume that

$$\lim_{h_1,h_2\to 0} \frac{k(x + h_1, x + h_2) - k(x + h_1, x) - k(x, x + h_2) + k(x, x)}{h_1 h_2}$$

exists.

a) Show that $\partial_x k(x_0, \cdot) \in \mathcal{N}_k$ for any $x_0 \in \Omega$. **Hint:** Consider the sequence $f_n(y) = \frac{k(x_0+h_n,y)-k(x_0,y)}{h_n}$ for $h_n \to 0$.

b) Show that any function $g \in \mathcal{N}_k$ has a first derivative g' and $g'(x_0) = \langle g, \partial_x k(x_0, \cdot)\rangle$.

Solution.

a) For a sequence $(h_n)_{n\in\mathbb{N}}$ with $h_n \to 0$, consider

$$f_n(y) = \frac{k(x + h_n, y) - k(x, y)}{h_n} \in \mathcal{N}_k.$$

Obviously, we have pointwise $\lim_{n\to\infty} f_n(y) = \partial_x k(x, y)$. Thus, we have to show that $f_n \to \partial_x k(x, \cdot)$ also in $\mathcal{N}_k$.

By assumption,

$$\lim_{n,m\to\infty} \langle f_n, f_m\rangle_k = \lim_{n,m\to\infty} \frac{k(x + h_n, x + h_m) - k(x + h_n, x) - k(x, x + h_m) + k(x, x)}{h_n h_m} = c$$

for some $c \geq 0$. But then

$$\lim_{n,m\to\infty} \|f_n - f_m\|_k^2 = \lim_{n,m\to\infty} \langle f_n, f_n\rangle_k - 2\langle f_n, f_m\rangle_k + \langle f_m, f_m\rangle_k = 0$$

and $(f_n)_{n\in\mathbb{N}}$ is Cauchy sequence in $\mathcal{N}_k$ with limit $f^* = \lim_{n\to\infty} f_n \in \mathcal{N}_k$. But then

$$f^*(y) = \langle f^*, k(y, \cdot)\rangle_k = \lim_{n\to\infty} \langle f_n, k(y, \cdot)\rangle_k = \lim_{n\to\infty} f_n(y) = \partial_x k(x, y).$$

b) Since $|\langle g, \partial_x k(x, \cdot)\rangle_k| \leq \|g\|_k \, \|\partial_x k(x, \cdot)\|_k < \infty$, the limit

$$\left|\lim_{n\to\infty} \frac{g(x_0 + h_n) - g(x_0)}{h_n}\right| = |\lim_{n\to\infty} \langle g, f_n\rangle_k| = |\langle g, \partial_x k(x_0, \cdot)\rangle|$$

exists. Consequently, $g'(x_0)$ exists and is simply given by $g'(x_0) = \langle g, \partial_x k(x_0, \cdot)\rangle$.

□