Exercise sheet 9 To be handed in on **Thursday, 23.06.2016**

Generalized interpolation, integration

1 Group exercises

G 1. (Integration in kernel spaces)

Let Ω be some set and $\rho : \Omega \to [0, \infty)$ a density function. Further let $\mathcal{H}$ be a Hilbert space of real-valued functions on Ω with kernel k such that $\int_\Omega \sqrt{k(x,x)}\rho(x)dx < \infty$.

a) Show that $\mathcal{H}$ consists of ρ-integrable functions and that the integration functional $\mathrm{Int}(f) = \int_\Omega f(x)\rho(x)dx$ is continuous.

b) Determine the function $h \in \mathcal{H}$ that represents the integration functional Int.

G 2. (Representation of general bounded linear functionals)

Let k be a kernel and $\mathcal{H}$ its native Hilbert space. Let $\lambda, \mu \in \mathcal{H}^*$ be two continuous functionals. Show that $\lambda^2 k(\cdot, y) \in \mathcal{H}$ and

$$\lambda(f) = \langle f, \lambda^2 k(\cdot, y)\rangle_k \qquad \text{for all } f \in \mathcal{H}.$$

Moreover, show that

$$\langle \lambda, \mu \rangle_{\mathcal{H}^*} = \lambda^1 \mu^2 k(x, y).$$

G 3. (Differentiable kernels)

Let $\Omega = [0, 1]$ and $k(x, y)$ be a kernel on Ω^2 which has continuous partial derivatives $\partial_x k$, $\partial_y k$. Assume that

$$\lim_{h_1,h_2 \to 0} \frac{k(x+h_1, x+h_2) - k(x+h_1, x) - k(x, x+h_2) + k(x, x)}{h_1 h_2}$$

exists.

a) Show that $\partial_x k(x_0, \cdot) \in \mathcal{N}_k$ for any $x_0 \in \Omega$. **Hint:** Consider the sequence $f_n(y) = \frac{k(x_0+h_n,y)-k(x_0,y)}{h_n}$ for $h_n \to 0$.

b) Show that any function $g \in \mathcal{N}_k$ has a first derivative g' and $g'(x_0) = \langle g, \partial_x k(x_0, \cdot)\rangle$.

2 Homework

H 1. (Worst-case integration error in kernel spaces)

Consider the same setting as in G1. For given points $x_1, \dots, x_n \in \Omega$ consider the *quadrature rule* $Q(f) = \frac{1}{n}\sum_{i=1}^{n} f(x_i)$ for $f \in \mathcal{H}$. The *worst-case integration error* of the quadrature rule Q in $\mathcal{H}$ is defined to be

$$e(Q, \mathcal{H}) := \sup_{f \in \mathcal{H}, \|f\|_k \le 1} |\mathrm{Int}(f) - Q(f)|.$$

a) Determine the function $\xi_Q \in \mathcal{H}$ that represents the integration error, that is, for $f \in \mathcal{H}$ we have $\langle \xi_Q, f \rangle_k = \mathrm{Int}(f) - Q(f)$. Then, show that $e(Q, \mathcal{H}) = \|\xi_Q\|_k$.

b) Derive a formula for $\|\xi_Q\|_k^2$ which depends only on the kernel k and the integration points $x_1, \dots, x_n$.

(5 Punkte)

H 2. (Generalized Interpolation)

Let $\mathcal{H}$ be a Hilbert space and $\lambda_1, \dots, \lambda_n \in H^*$ be linearly independent, continuous functionals with Riesz representers $v_1, \dots, v_n$ (**note:** we do not require $\mathcal{H}$ to be a reproducing kernel Hilbert space). Assume to be given $y_1, \dots, y_n \in \mathbb{R}$ and consider the *generalized interpolant*

$$\hat{f} = \mathrm{argmin}_{f \in \mathcal{H}: \lambda_j(f) = y_j} \|f\|.$$

Show that $\hat{f}$ is uniquely determined and $\hat{f} \in \mathrm{span}\{v_1, \dots, v_n\}$.

Hint 1: First show that there is an interpolant in $\mathrm{span}\{v_1, \dots, v_n\}$, then show that this interpolant has minimal norm. Finally, prove that the solution is unique.

Hint 2: In the end, the arguments are exactly the same as if the λ_i were function evaluations.

(7 Punkte)

H 3. (Differentiable kernels (cont.))

Let $\Omega = [0, 1]$ and k be a kernel that has continuous partial derivatives $\partial_{x^{\alpha_1}} \partial_{y^{\alpha_2}} k$ for $\alpha_1, \alpha_2 \in \mathbb{N}_0$ and $\alpha_1, \alpha_2 \le r$ for some $r \in \mathbb{N}$. The goal is to show that every $f \in \mathcal{N}_k$ has all derivatives up to order r and

$$f^{(m)}(x_0) = \langle f, \partial_{x^m} k(x_0, \cdot) \rangle_k, \qquad m \in \mathbb{N}_0, m \le r, \quad x_0 \in \Omega.$$

To this end, proceed as follows:

a) For $r = 1$, prove the statement by showing that the assumptions of G3 are fulfilled.
Hint: Fundamental theorem of calculus.

b) For $r > 1$, prove the statement by induction.

(8 Punkte)