Hierarchical Matrices

Summer semester 2013
Prof. Mario Bebendorf
Jos Gesenhues

universitätbonn

Exercise Sheet 1.

Due date: **Tuesday, 23.04.**

Exercise 1. (Block matrix rank)

Let $A \in \mathbb{C}^{m+n\times m+n}$ be regular and let

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where $A_{11} \in \mathbb{C}^{m\times m}$, $A_{22} \in \mathbb{C}^{n\times n}$. Furthermore let

$$A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

where $B_{11} \in \mathbb{C}^{m\times m}$, $B_{22} \in \mathbb{C}^{n\times n}$.
Prove that $\mathrm{rank}(\mathrm{B}_{12}) = \mathrm{rank}(\mathrm{A}_{12})$ and $\mathrm{rank}(\mathrm{B}_{21}) = \mathrm{rank}(\mathrm{A}_{21})$.

Exercise 2. (Low rank update for the inverse)

Let $A \in \mathbb{C}^{n\times n}$ be regular and $U, V \in \mathbb{C}^{n\times k}$.
Proof that $\det(A + UV^H) = \det(A)\det(I + V^H A^{-1} U)$ and conclude that $A + UV^H$ is regular iff $I + V^H A^{-1} U$ is regular.
Give a low rank update formula for the inverse of $A + UV^H$.

Exercise 3. (Maximum entry of a low rank matrix)

Let $U = [u_1, \dots, u_k], V = [v_1, \dots, v_k] \in \mathbb{C}^{n\times k}$. The object of this exercise is to find the maximum entry (in modulus) of UV^H without computing every entry $(UV^H)_{ij}$.
Let

$$C := \sum_{\ell=1}^{k} \mathrm{diag}(u_\ell) \otimes \mathrm{diag}(v_\ell)$$

where $\otimes$ is the Kronecker product. Find the eigenvalue λ_{ij} of C corresponding to the eigenvector $(e_i \otimes e_j)$ where e_i, e_j are unit vectors. To compute the maximal eigenvalue the power iteration can be used. To exploit the structure of C, the starting vector has the form $x \otimes y$ where $x, y \in \mathbb{C}^n$. Show that $C(x \otimes y)$ has in general a higher "rank" than $x \otimes y$. What can be done to overcome this?