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Practical Lab
Variational Methods and Inverse Problems in Imaging
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Depth from stereo – Problem sheet 4 — June 17th, 2014

At fist we will consider the ROF model, i.e. we aim for minimizing $E[u] = F[u] + G[\Lambda u]$, where $F[v] = \frac{\alpha}{2}\int_\Omega (v-g)^2 \,\mathrm{d}x$, $\Lambda v = \nabla v$, $G[q] = \int_\Omega |q| \,\mathrm{d}x$. We will however consider the three dimensional case, i.e. $\Omega \subset \mathbb{R}^3$, as we will later embed a lifted model for stereo reconstruction into this framework. Details will be presented in the lecture. In the following you will find a rough sketch on how to tackle the ROF model.

To find a minimizer for the ROF model we will employ a primal dual method, i.e. we will alternately minimize a primal and a dual energy. In the lecture you have already seen the conjugate. The problem is that E^* in our case is not easy to compute. However one can find a so called pre dual D that fulfills $E = D^*$. It is given by:

$$D[q] = \frac{1}{2\alpha}\|\mathrm{div}q + \alpha g\|^2 - \frac{\alpha}{2}\|g\|^2 + I_{K_1}[q], \quad \text{where } I_{K_1}[q] = \begin{cases} 0 & \text{if } |q| \leq 1 \\ \infty & \text{else} \end{cases}$$

Minimizers u of E and p of D will form a saddle point of the functional

$$\mathcal{L}[v,q] = \frac{\alpha}{2}\|v-g\|^2 - (v, \mathrm{div}q) - I_{K_1}[q]\,.$$

This saddle point problem can be tackled using an efficient gradient descent / ascent approach which alternately solves the following problems

$$\left(-\partial_\tau p_h^{n+1} + \nabla_h \tilde{u}_h^{n+1}, q_h - p_h^{n+1}\right)_h \leq 0 \quad \forall q_h \in \mathcal{V}_h^d \tag{p}$$

$$\partial_\tau u_h^{n+1} = -\alpha(u_h^{n+1} - g) + \mathrm{div} p_h^{n+1}, \quad u_h \in \mathcal{V}_h \tag{u}$$

where $\partial_\tau a^{n+1} = \frac{1}{\tau}(a^{n+1} - a^n)$ and $\tilde{u}_h^{n+1} = u_h^n + \tau\partial_\tau u_h^n = 2u_h^n - u_h^{n-1}$.

We will solve these problems on three dimensional simplicial grids and use piecewise affine finite elements as the ansatz space $\mathcal{V}_h$.

Problem (p) can be solved point wise explicitly, i.e. for every node $z \in \mathcal{N}_h$ the following update formula can be used.

$$p_h^{n+1} = \mathrm{Proj}_{\mathcal{K}}\left(p_h^n + \tau\nabla_h \tilde{u}_h^{n+1}\right), \quad \text{where } \mathrm{Proj}_{\mathcal{K}}(p) = \frac{p}{\max\{1, |p|\}}$$

Problem (u) will be solved in the usual finite element setting:

$$\left((1+\alpha)u_h^{n+1}, v_h\right)_h = (u_h^n, v_h)_h - \tau\left(p_h^{n+1}, \nabla_h v_h\right)_h + \alpha(g, v_h)_h$$

The scalar product $(\cdot,\cdot)_h$ represents lumped masses, i.e.

$$(q_h, p_h)_h = \int_\Omega \mathcal{I}_h[q_h \cdot p_h] \, \mathrm{d}x = \sum_{z \in \mathcal{N}_h} \left(\int_\Omega \phi_z \, \mathrm{d}x \right) p_h(z) \cdot q_h(z)$$

This approach will lead to a diagonal mass matrix on the left hand side whose inverse can be computed and applied efficiently.

Tasks:

- Go through the sketched algorithm and think about which kind of operators you will need to implement to tackle problems (u) and (p).
- Use your projection framework to implement $\mathrm{Proj}_{\mathcal{K}}$.
- Implement the finite element scheme to solve problem (u).