Scientific Computing II

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Lecturer: Prof. Dr. Beuchler
Assistent: Bastian Bohn

universitätbonn

Excercise sheet 11.

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Theoretical exercise 1. (Multigrid as preconditioner [15 points])

The multigrid algorithm can also be used to construct a preconditioner for an iterative solver of a FE-system. Let $K_l \in \mathbb{R}^{N_l \times N_l}$ denote the stiffness matrix on level l, let $k \in \mathbb{N}$ be the number of iterations of the multigrid cycle and let $\mu \in \mathbb{N}$ describe the specific multigrid method (i.e. $\mu = 1$ for V-cycle, $\mu = 2$ for W-cycle, etc.). Then the matrix of the corresponding preconditioning operator on level L can be written as

$$C_L^{-1} := (\mathrm{id}_L - (M_L)^k) K_L^{-1}, \tag{1}$$

where $\mathrm{id}_l \in \mathbb{R}^{N_l \times N_l}$ denotes the identity matrix and $M_l \in \mathbb{R}^{N_l \times N_l}$ is the multigrid operator defined by

$$\begin{aligned} M_2 &:= S_2^{\mathrm{post}}(\mathrm{id}_2 - I_1^2 K_1^{-1} I_2^1 K_2) S_2^{\mathrm{pre}}, \\ M_l &:= S_l^{\mathrm{post}}(\mathrm{id}_l - I_{l-1}^l(\mathrm{id}_{l-1} - M_{l-1}^{\mu}) K_{l-1}^{-1} I_l^{l-1} K_l) S_l^{\mathrm{pre}}, \quad l = 3, \dots, L \end{aligned}$$

with prolongation matrices I_{l-1}^l and restriction matrices $I_l^{l-1} = (I_{l-1}^l)^T$. For some smoothing parameter $\omega_l \in \mathbb{R}^+$ and a fixed number of pre- and post-smoothing steps ν_{pre} and ν_{post}, the matrices for the pre- and post-smoothing steps can be written as

$$\begin{aligned} S_l^{\mathrm{pre}} &:= (\mathrm{id}_l - \omega_l A_l^{-1} K_l)^{\nu_{\mathrm{pre}}} \\ S_l^{\mathrm{post}} &:= (\mathrm{id}_l - \omega_l (A_l^{-1})^T K_l)^{\nu_{\mathrm{post}}}, \end{aligned}$$

In the following assume that $\nu_{\mathrm{pre}} = \nu_{\mathrm{post}}$.

a) Prove that M_l is the error propagation operator for the multigrid method, i.e.

$$M_l(u - u_l^{(j)}) = u - u_l^{(j+1)},$$

where $u_l^{(j)}$ is the j-th iterate of the solution and u is the solution of $K_l u = f_l$. To this end, recall the definition of iterative methods.

b) Give A_l for pre- and post-smoothing steps for Jacobi- and for Gauß-Seidel smoother. What is the difference between the pre- and post-smoother for Gauß-Seidel?

c) Prove that $\langle S_l^{\mathrm{pre}} u, v\rangle_{K_l} = \langle u, S_l^{\mathrm{post}} v\rangle_{K_l}$, where $\langle u, v\rangle_A := \langle Au, v\rangle$ for s.p.d. A.

d) Prove that $\langle M_2 u, v\rangle_{K_2} = \langle u, M_2 v\rangle_{K_2}$

e) Prove that $\langle M_l u, v\rangle_{K_l} = \langle u, M_l v\rangle_{K_l}$ for $2 < l \leq L$.

f) Prove that (1) is symmetric.

Theoretical exercise 2. (Discrete norms [5 points])

For a s.p.d. matrix $A \in \mathbb{R}^{n \times n}$ let the discrete norm $\|\!|\cdot|\!\|_{s,A}$ be defined by

$$|\!|\!| x |\!|\!|_{s,A}^2 := \langle x, A^s x \rangle$$

for some $s \in \mathbb{R}$. Assume there exists an $\alpha > 0$ such that $\langle x, Ax \rangle \geq \alpha \langle x, x \rangle$. Prove that

$$\alpha^{-\frac{t}{2}} |\!|\!| x |\!|\!|_{t,A} \geq \alpha^{-\frac{s}{2}} |\!|\!| x |\!|\!|_{s,A}$$

for $t \geq s$.