Scientific Computing II

Summer Semester 2014
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Excercise sheet 6.

Closing date **20.05.2014**.

Theoretical exercise 1. (L_∞ error estimator [5 points])

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and connected domain with Lipschitz boundary and cone condition with $x_0 \in \Omega$. Furthermore, let $a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx$ for $u, v \in H_0^1(\Omega)$. Let $\rho_0 > 0$ be arbitrary, and let $\delta \in C_0^\infty(\Omega)$ be such that there for a constant $C > 0$:

$$\begin{aligned} \int_\Omega \delta(x) dx &= 1 \\ 0 \leq \delta(x) &\leq C\rho_0^{-2} \quad \forall\, x \in \Omega \\ \operatorname{supp}(\delta) &\subset B_0 := \{x \in \Omega \mid \|x - x_0\| \leq \frac{\rho_0}{2}\} \end{aligned}$$

With the standard notations from the lecture, prove that

$$a(G, u - u_h) = \sum_{s=1}^{\#\text{elements}} \int_{\tau_s} (G - I_h G) r_h dx + \sum_{r=1}^{\#\text{edges}} \int_{e_r} R_h (G - I_h G) ds,$$

where G is the regularized Green's function to the regularized delta function δ and I_h is the interpolation operator.

Theoretical exercise 2. (Linear combination of bubble functions [5 points])

For a uniform FE triangulation on a domain Ω as in exercise 1, let $\tilde{\tau}$ be the patch to the element τ and set

$$v(x) = \sum_{\tau_s \subset \tilde{\tau}} \alpha_s \phi_{I,s}(x) + \sum_{e_r \subset \partial\tau} \beta_r \phi_{O,r}(x)$$

where $\phi_{I,s}$ is the interior bubble function of the element τ_s and $\phi_{O,r}$ is the edge bubble function corresponding to e_r. The coefficients α_s, β_r are defined via

$$\begin{aligned} \int_{\tau_s} v \operatorname{sgn}(\overline{r}) dx &= h_{\tau_s}^2 \quad \forall\, \tau_s \in \tilde{\tau} \\ \int_{e_r} v \operatorname{sgn}(\overline{R}) ds &= \begin{cases} h_\tau & \text{if } e_r \subset \partial\tau \setminus \partial\Omega \\ 0 & \text{else} \end{cases} \end{aligned}$$

where h_{τ_s} is the diameter of τ_s and $\overline{r}, \overline{R}$ denote piecewise constant approximations to the interior residuals and the boundary residuals. Prove that the coefficients α_s, β_r (for $\{r \mid e_r \not\subseteq \partial\Omega\}$) are uniformly bounded (without dependence of the constants on the element diameters) from above and below.

Theoretical exercise 3. (Regularized Green's functions [**Bonus**: 5 points])

For the setting from exercise 1 assume additionally that there exists $1 < p_0 \leq \frac{4}{3}$ such that

$$\|G\|_{W^{2,p_0}(\Omega)} \leq \tilde{C} \|\delta\|_{L_{p_0}(\Omega)}.$$

Prove that there exists a $C > 0$ independent of p and ρ_0 such that

$$|G|_{W^{2,p}(\Omega)} \leq \frac{C\rho_0^{-4(p-1)}}{(p-1)^2}.$$

for $p \downarrow 1$.
Hint: You may assume that there exists a $c > 0$ such that

$$\|G\|_{L_{\frac{p}{p-1}}(\Omega)} \leq \frac{c}{(p-1)^{\frac{1}{2}}}\|\nabla G\|_{L_2(\Omega)} \leq \frac{c}{p-1}\|G\|_{L_p(\Omega)}$$

for $p \downarrow 1$. This follows from Sobolev inequalities.