Scientific Computing II

Summer Semester 2014
Lecturer: Prof. Dr. Beuchler
Assistent: Bastian Bohn

universitätbonn

Excercise sheet 9.

Closing date **17.06.2014**.

Theoretical exercise 1. (Matrix bounds [5 points])

Let K and C be symmetric, positive definite matrices of the same size. Let furthermore

$$c_1 C \leq K \leq c_2 C$$

and $\underline{u}, \underline{\tilde{u}}$ be the solution of $K\underline{u} = \underline{b}$ and $C\underline{\tilde{u}} = \underline{b}$, respectively.
Prove that

$$\frac{1}{c_2^2}\langle K\underline{\tilde{u}}, \underline{\tilde{u}}\rangle \leq \langle K\underline{u}, \underline{u}\rangle \leq \frac{1}{c_1^2}\langle K\underline{\tilde{u}}, \underline{\tilde{u}}\rangle \quad \forall \underline{u} \leftrightarrow \underline{\tilde{u}} \in \mathbb{R}^N.$$

Theoretical exercise 2. (Positive definiteness of additive Schwarz preconditioners [5 points])

Prove that the matrix representation of an additive Schwarz preconditioner matrix is always positive definite (given that the matrix which is used originally (e.g. stiffness matrix) is positive definite).

Theoretical exercise 3. (Computational Cost of the BPX application [5 points])

Prove that for the Poisson model problem in some dimension $d > 0$ discretized by linear Lagrange FE, the matrix vector multiplication $C^{-1}r = w$ for the BPX preconditioning matrix C^{-1} can be computed in $\mathcal{O}(N_L)$ operations, where N_L is the number of basis functions on the highest level L.

Theoretical exercise 4. (BPX and diagonal scaling [5 points])

We defined the BPX-preconditioner for a second order elliptic PDE by

$$C^{-1} := \sum_{l=0}^{L} I_l^L D_l^{-1} I_L^l,$$

where I_l^L is the prolongation/interpolation matrix from level l to L, $I_L^l = (I_l^L)^T$ and D_l is the diagonal of the stiffness matrix on level l. The original definition of the BPX preconditioner is a little bit different: Prove that for dimensions $d = 2$ and $d = 3$, for a second order elliptic PDE with corresponding coercive and bounded symmetric bilinear form $a(\cdot, \cdot)$ both preconditioners

a)

$$\tilde{C}^{-1} := \sum_{l=0}^{L} I_l^L I_L^l \cdot \begin{cases} 1 & \text{for } d = 2 \\ h_l^{-1} & \text{for } d = 3 \end{cases}$$

b)

$$\tilde{C}^{-1} := \sum_{l=0}^{L} I_l^L \tilde{D}_l I_L^l \;\text{ with diagonal matrix } (\tilde{D}_l)_{ii} := \begin{cases} a(\phi_i^L, \phi_i^L) & \text{for } d = 2 \\ a(\phi_i^L, \phi_i^L)\left(\frac{h_l}{h_L}\right)^{-1} & \text{for } d = 3 \end{cases}, \tag{1}$$

where ϕ_i^L denotes the basis function on level L settled in the i-th node (i.e. the same node as ϕ_i^l on level l),

yield condition numbers $\kappa(\tilde{C}^{-1}K_h)$ of the same order (in $\mathcal{O}$-notation) as $\kappa(C^{-1}K_h)$, where K_h is the stiffness matrix on the finest level.