Numerical Simulation

Summer Semester 2015
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Excercise sheet 1.

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Theoretical exercise 1. (Convex sets and differentiability [5 points])

Let $X \subset \mathbb{R}^d$ be open and convex. Let furthermore $f : X \to \mathbb{R}$ be continuously differentiable. Prove that f is convex if and only if

$$f(x) - f(y) \geq (\nabla f(y))^T (x - y) \quad \forall\, x, y \in X, x \neq y.$$

Theoretical exercise 2. (Lagrange multipliers [5 points])

Consider the constrained minimization problem

$$\min_{(x,y)\in\mathbb{R}^2} f(x,y) := 3x^2 + y^2 \quad \text{such that} \quad g(x,y) := \frac{3}{2}x^2 + y = 2.$$

a) Write down the Lagrange function for this problem and solve the constrained minimization problem.

b) Draw the contour lines of f for the values $f(x,y) = 1$, $f(x,y) = 3$ and $f(x,y) = 12$ and the contour line of g for the value $g(x,y) = 2$. Give a geometrical interpretation for the method of Lagrange multipliers.

Theoretical exercise 3. (Lower semicontinuity [7 points])

Let X be a Banach space and let $M \subset X$. Recall that $F : M \to [-\infty, \infty]$ is *sequentially lower semicontinuous* at $x \in M$ iff for each sequence $(x_n)_{n=1}^{\infty}$ in M with $x_n \overset{n\to\infty}{\longrightarrow} x$ it holds

$$F(x) \leq \liminf_{n\to\infty} F(x_n).$$

We call F sequentially lower semicontinuous on M iff F is sequentially lower semicontinuous on every $x \in M$.
Furthermore we call F *lower semicontinuous* iff

$$\{y \in M \mid F(y) \leq r\}$$

is closed (relative to M) for all $r \in \mathbb{R}$.
Prove the following:

a) F is sequentially lower semicontinuous on $M \Leftrightarrow F$ is lower semicontinuous on M.

b) Let F,G,$(F_\alpha)_{\alpha\in I} : M \to [-\infty, \infty]$ be sequentially lower semicontinuous for some index set I. Assume that $F + G$ is well-defined then $F + G$, $\sup(F, G)$, $\inf(F, G)$, $\sup_{\alpha\in I} F_\alpha$ are sequentially lower semicontinuous. If furthermore $F, G \geq 0$, then $F \cdot G$ is also sequentially lower semicontinuous.

Theoretical exercise 4. (Uniqueness of weak limits [3 points])

Let X be a Banach space. Assume that $x, y \in X$ and $x_n \in X \;\forall n \in \mathbb{N}$. Prove that $x = y$ if $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$.