Numerical Simulation

Summer Semester 2015
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Excercise sheet 10.

Closing date **30.06.2015**.

Theoretical exercise 1. (The space $L^p(0,T;X)$ [6 points])

Let X, Y be real Banach spaces and let $1 \leq p \leq \infty$. Let furthermore $0 < T < \infty$.

a) Show that the embedding $C([0,T],X) \hookrightarrow L^p(0,T;X)$ is continuous.

b) Let $X \subseteq Y$ and let the corresponding embedding be continuous. Prove that the embedding $L^p(0,T;X) \hookrightarrow L^q(0,T;Y)$ is continuous for all $1 \leq q \leq p \leq \infty$.

c) Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Prove that the embedding $L^q(0,T;X^*) \hookrightarrow L^p(0,T;X)^*$ is continuous with embedding constant 1.

Theoretical exercise 2. (The space $W^{1,p}(0,T;V,H)$ [6 points])

Let $1 < p < \infty$ and let $V \subseteq H \subseteq V^*$ be a Gelfand triple. Let furthermore $0 < T < \infty$. We define the space $C^1([0,T];V)$ as the Banach space of continuous functions $f : [0,T] \to V$ such that

$$\|f\|_{C^1([0,T];V)} := \max_{0 \leq t \leq T} \|f(t)\|_V + \max_{0 \leq t \leq T} \|f'(t)\|_V < \infty.$$

a) Prove that for $u, v \in C^1([0,T];V)$ it holds

$$\langle u(t), v(t)\rangle_H - \langle u(s), v(s)\rangle_H = \int_s^t \langle u'(z), v(z)\rangle_{V^*,V} + \langle v'(z), u(z)\rangle_{V^*,V} \, \mathrm{d}z,$$

where $\langle \cdot, \cdot \rangle_{V^*,V}$ is the natural duality pairing for V^* and V.

b) Prove that there exists a continuous embedding

$$W^{1,p}(0,T;V,H) \hookrightarrow C([0,T],H).$$

Theoretical exercise 3. (Example function [4 points])

Prove that $y(t,x) = \frac{e^t}{\sqrt{x}}$ is an element of $C([0,1], L^1([0,1]))$ and calculate its norm. For which $1 \leq p \leq \infty, 1 \leq q \leq \infty$ is $y \in L^p(0,1; L^q([0,1]))$?