Numerical Simulation

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Excercise sheet 11.

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Theoretical exercise 1. (Enhanced minimization functional [6 points])

For $W(0,T) := W^{1,2}(0,T;H^1(\Omega),L^2(\Omega))$, consider the model problem of optimal boundary control with enhanced minimization functional:

$$\min_{u\in L^2(\Sigma),y\in W(0,T)} \frac{1}{2}\|y(\cdot,T)-y_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L^2(\Sigma)} + \iint\limits_Q a_Q y \, \mathrm{d}x\mathrm{d}t + \iint\limits_\Sigma u_\Sigma u \, \mathrm{d}s\mathrm{d}t$$

with $y_\Omega \in L_2(\Omega), a_Q \in L^2(Q), u_\Sigma \in L^2(\Sigma)$ for $Q = \Omega \times (0,T)$ with Lipschitz domain Ω and $\Sigma = \partial\Omega \times (0,T)$ such that

$$\begin{aligned} y_t - \Delta y &= 0 && \text{in } Q \\ \partial_\nu y + \alpha y &= \beta u && \text{on } \Sigma \\ y(\cdot,0) &= y_0(\cdot) && \text{in } \Omega \end{aligned}$$

and $u_a \le u \le u_b$ almost everywhere for $u_a, u_b \in L^2(\Sigma)$. Here, $y_0 \in L^2(\Omega)$, $\beta \in L^\infty(\Sigma)$ and $\alpha \in L^\infty(\Sigma)$ is non-negative almost everywhere.
Prove that also for this problem an optimal control exists and derive the corresponding variational inequality and the adjoint equation.

Theoretical exercise 2. (Monotone Lipschitz operators [6 points])

Let V be a real Hilbert space and $A : V \to V^*$ be a strongly monotone and Lipschitz-continuous operator, i.e.

$$\exists\, \beta_0 > 0 : \quad \langle Au - Av, u - v\rangle_{V^*,V} \ge \beta_0 \|u - v\|_V^2$$

and

$$\exists\, L > 0 : \quad \|Au - Av\|_{V^*} \le L\|u - v\|_V$$

for all $u, v \in V$.
Prove that for every $f \in V^*$ there exists exactly one $v \in V$ such that $Av = f$, i.e. $A^{-1} : V^* \to V$ exists. Prove furthermore that A^{-1} is Lipschitz continuous with Lipschitz constant $\frac{1}{\beta_0}$.

Theoretical exercise 3. (Fréchet differentiability in L_p [4 points])

Let $\Phi : L^p(0,1) \to L^q(0,1)$ be defined by $\Phi(f)(\cdot) = \sin(f(\cdot))$. Prove that Φ is not Fréchet differentiable for $1 \le p \le q < \infty$.