Numerical Simulation

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Excercise sheet 5.

Closing date **19.05.2015**.

Theoretical exercise 1. (Optimization in $L_2([0,1])$ [4 points])

Determine all solutions to the minimization problem

$$\min \int_0^1 \left(u^2(x) - 1\right)^2 \mathrm{d}x \quad \text{s.t. } |u(x)| \leq 1 \text{ a.e. and } u \in L_2([0,1]).$$

Find a Banach space in which all different solutions have the same distance to each other.

Theoretical exercise 2. (Optimal control: Poisson equation [10 points])

Let Ω be a bounded Lipschitz domain and let $\lambda > 0$ and consider $u, y_\Omega, u_a, u_b \in L_2(\Omega)$ and $\beta \in L_\infty(\Omega), y \in H_0^1(\Omega)$. Let us define the optimal control problem

$$\min J(y,u) := \frac{1}{2}\|y - y_\Omega\|^2_{L_2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L_2(\Omega)}$$

with side conditions

$$\begin{aligned} -\Delta y &= \beta u && \text{in } \Omega \\ y &= 0 && \text{on } \partial\Omega \\ u &\in U_{ad} := \{v \mid u_a \leq v \leq u_b \text{ almost everywhere on } \Omega\}. \end{aligned}$$

Prove that (y,u) is a solution to the control problem if and only if

$$\begin{aligned} -\Delta y &= \beta u && \text{in } \Omega \\ y &= 0 && \text{on } \partial\Omega \\ u &\in U_{ad} && \\ \langle \beta p + \lambda u, v - u\rangle_{L_2(\Omega)} &\geq 0 && \forall\, v \in U_{ad} \end{aligned}$$

where p is given by

$$\begin{aligned} -\Delta p &= y - y_\Omega && \text{in } \Omega \\ p &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Theoretical exercise 3. (Interior points in L_p [4 points])

Let $1 \leq p < \infty$ and let $a, b \in L_\infty([0,1])$. Prove that the set

$$\{u \in L_p([0,1]) \mid a(x) \leq u(x) \leq b(x) \text{ for almost every } x \in [0,1]\}$$

has no interior points.