Numerical Simulation

Summer Semester 2015
Lecturer: Prof. Dr. André Uschmajew
Assistent: Bastian Bohn

universitätbonn

Excercise sheet 6.

Closing date **02.06.2015**.

Theoretical exercise 1. (Generic elliptic PDEs [7 points])

Let Ω be a bounded d-dimensional Lipschitz domain where the boundary is the disjoint union $\Gamma := \partial\Omega = \Gamma_0 \cup \Gamma_1$. Let $c_0 \in L_\infty(\Omega), \alpha \in L_\infty(\Gamma_1)$ fulfill $c_0 \geq 0$ a.e. in Ω and $\alpha \geq 0$ a.e. on Γ_1. Assume that one of the following conditions holds:

(i) $|\Gamma_0| > 0$,

(ii) $\Gamma_1 = \Gamma$ and $\int_\Omega c_0^2(x)\mathrm{d}x + \int_\Gamma \alpha^2 \mathrm{d}s(x) > 0$.

Let furthermore $f \in L_2(\Omega)$ and $g \in L_2(\Gamma_1)$ and consider the following problem

$$\begin{aligned} \mathcal{A}u + c_0 u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma_0 \\ \partial_{\nu_\mathcal{A}} u + \alpha u &= g && \text{on } \Gamma_1, \end{aligned}$$

where $u \in V := \{u \in H^1(\Omega) \mid u|_{\Gamma_0} = 0\}$. The divergence operator $\mathcal{A}$ is defined by $\mathcal{A}u := -\mathrm{div}\,(A\,\mathrm{grad}(u))$ for a symmetric matrix A whose entries are $L_\infty(\Omega)$ functions. Let $\gamma_0 > 0$ be such that

$$y^T A y \geq \gamma_0 \|y\|_{\ell_2}^2 \quad \forall\, y \in \mathbb{R}^d$$

The conormal is defined via $\nu_\mathcal{A} := A\nu$ where ν is the outer normal on Γ_1.
Compute the weak formulation of the above PDE and show that the Lax-Milgram Lemma can be applied, i.e. show that there exists exactly one solution $u^* \in V$ which solves the PDE and there exists a constant $c > 0$ such that

$$\|u^*\|_{H^1(\Omega)} \leq c\left(\|f\|_{L_2(\Omega)} + \|g\|_{L_2(\Gamma_1)}\right).$$

Theoretical exercise 2. (Dual optimization for quadratic problems [5 points])

Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, $b \in \mathbb{R}^n, d \in \mathbb{R}^m$ and let $C \in \mathbb{R}^{m\times n}$ have rank $m \leq n$. Consider the quadratic minimization problem

$$\frac{1}{2}u^T A u + b^T u \to \min!_{u\in\mathbb{R}^n} \qquad \text{s.t. } Cu - d \leq 0,$$

where the inequality has to be understood componentwise. Assume that there exists a $u \in \mathbb{R}^n$ such that $Cu - d \leq 0$. Let $L : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be the corresponding Lagrangian. Prove that the dual maximization problem

$$G(\lambda) := \inf_{u\in\mathbb{R}^n} L(u,\lambda) \to \max!_{\lambda\in[0,\infty)^m}$$

has a unique solution λ^* and compute a closed form of $G(\lambda)$.

Theoretical exercise 3. (Bang-Bang optimality conditions [6 points])

Let Ω be a bounded Lipschitz domain with boundary Γ and let $y_\Omega, e_\Omega \in L_2(\Omega)$ and $e_\Gamma \in L_2(\Gamma)$. Assume that there exists $y_0 \in H^2(\Omega)$ such that the trace fulfills $\tau(y_0) = e_\Gamma$. Determine the optimality conditions of the control problem

$$\min_{y \in H^2(\Omega)} \int_\Omega (y(x) - y_\Omega(x))^2 \mathrm{d}x \quad \text{ s.t. } -\Delta y = u + e_\Omega,\ y|_\Gamma = e_\Gamma \ \text{ and } -1 \leq u(x) \leq 1 \text{ a.e.},$$

i.e. verify that Theorem 1.25 is applicable and compute the adjoint operator S^* to get a direct variational inequality for the optimal control u.