Numerical Simulation

Summer Semester 2015
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Excercise sheet 7.

Closing date **09.06.2015**.

Theoretical exercise 1. (Pointwise inequality contraints [5 points])

Let Ω be a bounded Lipschitz domain and let $u_a, u_b \in L_2(\Omega)$ fulfill $u_a \leq u_b$ almost everywhere. Prove that

$$U_{\text{ad}} := \{u \in L_2(\Omega) \mid u_a \leq u \leq u_b \text{ almost everywhere in } \Omega\}$$

is a convex, bounded and closed subset of $L_2(\Omega)$.

Theoretical exercise 2. (Convergence of the Uzawa algorithm [8 points])

Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite, $b \in \mathbb{R}^n, d \in \mathbb{R}^m$ and let $C \in \mathbb{R}^{m\times n}$ have rank $m \leq n$. Now consider the indefinite system

$$\begin{pmatrix} A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}. \tag{1}$$

The *Uzawa* algorithm is defined by

Algorithm 1: Uzawa algorithm

Data: $\lambda^{(0)} \in \mathbb{R}^m, \alpha > 0, \epsilon \geq 0$

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Set k := 1;
do
    Solve Au^(k) = b − C^T λ^(k−1);
    Set λ^(k) := λ^(k−1) + α(Cu^(k) − d);
    Set k := k + 1;
while ‖λ^(k) − λ^(k−1)‖ > ε;
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Prove that the sequence $(u^{(k)}, \lambda^{(k)})$ converges to the true solution (u, λ) of (1) if $\epsilon = 0$ and $\alpha < \frac{2}{\|S\|}$, where $S = CA^{-1}C^T$ denotes the Schur complement.

Theoretical exercise 3. (Optimal stationary heat equation with boundary control [5 points])

Let Ω be a bounded Lipschitz domain with boundary Γ and let $y_\Omega \in L_2(\Omega)$ and let $\alpha \in L_\infty(\Gamma)$ be a non-negative function with

$$\int_\Gamma \alpha^2 \mathrm{d}s > 0.$$

Determine the optimality conditions of the control problem

$$\min_{y \in H^1(\Omega), u \in L_2(\Gamma)} \|y(x) - y_\Omega(x)\|^2_{L_2(\Omega)} + \frac{\lambda}{2}\|u\|^2_{L_2(\Gamma)}$$

such that

$$\begin{array}{rcll} -\Delta y & = & 0 & \text{in } \Omega, \\ \partial_\nu y & = & \alpha(u-y) & \text{on } \Gamma, \\ -1 \leq & u(x) & \leq 1 & \text{a.e. on } \Gamma, \end{array}$$

where ν is the outer normal on Γ. Use the framework of section 2.4. of the lecture and check that the corresponding forms $a(y, v)$ und $F_u(v)$ are indeed bilinear and bounded. Derive the variational inequality for the adjoint state.