Wissenschaftliches Rechnen II
(Scientific Computing II)

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Exercise 1. *(product rule)*
Let $u, \partial_j u \in L_{q,\mathrm{loc}}(\Omega)$ and $v, \partial_j v \in L_{q',\mathrm{loc}}(\Omega)$ for $1 < q < \infty$ with $\frac{1}{q} + \frac{1}{q'} = 1$. Prove the product rule

$$\partial_j(uv) = (\partial_j u)v + u\partial_j(v).$$

Exercise 2. *(trace identity and inequality)*
Let $T := \mathrm{conv}(\{P\} \cup E) \subseteq \mathbb{R}^2$ be a triangle with edge E and opposite vertex P.

(a) Prove for any $f \in W^{1,1}(T)$ the trace inequality

$$\fint_E f\, ds = \fint_T f\, dx + \frac{1}{2} \fint_T \nabla f(x) \cdot (x - P)\, dx.$$

(b) Let $f \in H^1(T)$ and $h := \sup_{x \in E} |P - x|$. Prove the trace inequality

$$\|f\|^2_{L^2(E)} \leq \frac{|E|}{|T|} \|f\|_{L^2(T)} (\|f\|_{L^2(T)} + h \|\nabla f\|_{L^2(T)}).$$

(c) Let $f \in H^1(T)$. Prove that there exists a constant $C > 0$ independent of f, $|E|$, $|T|$ such that

$$\|f\|_{L^2(E)} \leq C(h^{-1/2} \|f\|_{L^2(T)} + h^{1/2} \|\nabla f\|_{L^2(T)}).$$

Exercise 3. *(Sobolev embedding $H^2(T) \hookrightarrow C^{0,1/2}(T)$ on a triangle)*
Let $T \subseteq \mathbb{R}^2$ be a triangle and $v \in H^2(T)$.

(a) Consider a sub-triangle $t := \mathrm{conv}\{A, B, C\}$ with $E := \mathrm{conv}\{A, B\}$ and with tangent vector τ. Apply the trace inequality to $f|_E := \nabla v \cdot \tau$ and prove that

$$|v(B) - v(A)| \leq |E|^{1/2} \varrho^{-1/2} 2 \big(1 + \mathrm{diam}(t)^2\big)^{1/2} \|v\|_{H^2(t)}$$

for $\varrho := 2|t|/|E|$.

(b) For any two points A and B in T there exists $C \in T$ such that (with $E := \mathrm{conv}\{A, B\}$ and $t := \mathrm{conv}\{A, B, C\}$), ϱ^{-1} is uniformly bounded by some constant $C(T)$ that depends only on T, but not on A, B, or t.

(c) Conclude that v is Hölder continuous with exponent $1/2$.

Exercise 4. *(nodal interpolation not L^2 or H^1 stable)*
For a triangle $T \subseteq \mathbb{R}^2$, prove that there is no constant C such that the nodal P_1 interpolation I satisfies

$$\|Iu\|_{L^2(T)} \leq C \|u\|_{L^2(T)} \text{ for all } u \in H^2(T)$$
$$\text{or} \quad \|\nabla Iu\|_{L^2(T)} \leq C \|\nabla u\|_{L^2(T)} \text{ for all } u \in H^2(T).$$

Exercise 5. *(maximal angle condition)*
Prove that the maximal angle condition is necessary for the interpolation error estimate

$$\exists C \text{ independent of } h_T \text{ such that } \forall u \in H^2(T) : |u - Iu|_{H^1(T)} \leq C h_T |u|_{H^2(T)}.$$