Wissenschaftliches Rechnen II
(Scientific Computing II)

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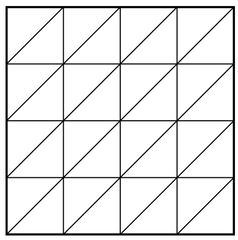
Exercise 34. *(Standard FEMs are unstable for Stokes)*
Let $\Omega = (0,1)^2$. Prove that the following discretizations of the Stokes equations lead to unstable saddle-point problems (prove that the discrete LBB condition is violated).

(a) $V_h := [P_0^{1,0}(\mathcal{T}_h)]^2$ and $M_h := P^{0,-1}(\mathcal{T}_h) \cap L_0^2(\Omega)$ on the criss triangulation $\mathcal{T}_h$.

Hint: Use a dimension argument. The criss triangulation is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(b) $V_h := [Q_0^{1,0}(\mathcal{T}_h)]^2$ und $M_h := P^{0,-1}(\mathcal{T}_h) \cap L_0^2(\Omega)$ for a uniform partition $\mathcal{T}_h$ of Ω in squares. Here

$$Q_0^{1,0}(\mathcal{T}_h) := \{v \in H_0^1(\Omega) \,|\, \forall T_h \in \mathcal{T} \, \exists (a,b,c,d) \in \mathbb{R}^4 : v|_T(x,y) = a + bx + cy + dxy\}$$

denotes the space of bilinear finite elements.

Hint: Find $q_h \in M_h$ with $\int_\Omega q_h \operatorname{div} v_h \, dx = 0$ for all $v_h \in V_h$.

Exercise 35. *(Discrete LBB-condition for Crouzeix-Raviart FEM)*
Let $\mathcal{T}_h$ be a regular triangulation of the bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^d$. Define the interpolation operator $I_{\mathrm{CR}} : H_0^1(\Omega) \to \mathrm{CR}_0^1(\mathcal{T}_h)$ by the condition

$$\forall v \in H_0^1(\Omega) \, \forall F \in \mathcal{F}_h \qquad (I_{\mathrm{CR}} v)(\mathrm{mid}(F)) = \fint_F v \, ds,$$

where $\mathcal{F}_h$ denotes the set of $(d-1)$-dimensional hyperfaces (edges in 2D, faces in 3D).

(a) Prove that I_{CR} is well-defined and that

$$\forall v \in H_0^1(\Omega) \, \forall F \in \mathcal{F}_h \qquad \int_F (v - I_{\mathrm{CR}} v) \, ds = 0.$$

(b) Prove that

$$\forall v \in H_0^1(\Omega) \, \forall T \in \mathcal{T}_h \qquad \int_T \nabla I_{\mathrm{CR}} v \, dx = \int_T \nabla v \, dx.$$

(c) Let $M_h := P^{0,-1}(\mathcal{T}_h) \cap L_0^2(\Omega)$. Prove that there exists a constant $\beta > 0$ independent of h such that

$$\beta \le \inf_{q_h \in M_h \setminus \{0\}} \sup_{v_h \in [\mathrm{CR}_0^1(\mathcal{T}_h)]^d \setminus \{0\}} \sum_{T \in \mathcal{T}_h} \frac{\int_T (\operatorname{div} v_h) \, q_h \, dx}{\|D_{\mathrm{NC}} v_h\|_{L^2(\Omega)} \|q_h\|_{L^2(\Omega)}}.$$

Here, $\|D_{\mathrm{NC}} v_h\|_{L^2(\Omega)} := \sqrt{\sum_{T \in \mathcal{T}_h} \|D v_h\|_{L^2(T)}^2}$ denotes the norm of the piecewise derivative of v_h.

Hint: Prove (c) by using the continuous LBB condition for the Stokes equations and (b).

Exercise 36. *(Euler's formulae)*
Let $\mathcal{T}$ be a regular triangulation of the simply-connected bounded domain $\Omega \subseteq \mathbb{R}^2$ with vertices $\mathcal{N}$, edges $\mathcal{E}$ and interior edges $\mathcal{E}(\Omega)$. Prove that

$$\#\mathcal{N} + \#\mathcal{T} = 1 + \#\mathcal{E}$$

and

$$2\,\#\mathcal{T} + 1 \; = \#\mathcal{N} + \#\mathcal{E}(\Omega)$$

($\#A$ denotes the cardinality of a set A).

Exercise 37. *(Basis of piecewise divergence-free Crouzeix-Raviart functions)*
Let $\mathcal{T}_h$ be a regular triangulation of the bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^2$. The space Z_{CR} of piecewise divergence-free Crouzeix-Raviart functions with boundary conditions is defined as

$$Z_{\mathrm{CR}} := \left\{ v_{\mathrm{CR}} \in [\mathrm{CR}^1_0(\mathcal{T}_h)]^2 \;\middle|\; \forall T \in \mathcal{T}_h \;\operatorname{div} v_{\mathrm{CR}}|_T = 0 \right\}.$$

Find a basis of Z_{CR}. Compare the result with Exercise 33.

Hint: Find suitable linear independent functions and use a dimension argument (Euler formulae from Exercise 36). You may assume that Ω is simply-connected.