Wissenschaftliches Rechnen II
(Scientific Computing II)

Sommersemester 2015
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Exercise 12. *(discontinuous Sobolev functions)*
Consider the unit sphere $\Omega := \{x \in \mathbb{R}^3 : |x| < 1\}$ and the function $u(x) := \log |x|$ for $x \in \Omega$. Prove

(a) $u \in L^2(\Omega)$, but $u \notin C^0(\Omega)$.

(b) $u \in H^1(\Omega)$.

Exercise 13. *(Laplacian in polar coordinates)*
Prove that in polar coordinates as the Laplacian reads as

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}.$$

Exercise 14. *(sector domain)*
Let $\omega \in (0, 2\pi)$ and $\alpha \in (0, 1)$. In polar coordinates (r, φ) the sector domain reads

$$\Omega_\omega := \{(r, \varphi) \in (0, 1) \times (0, 2\pi) : \phi < \omega\}.$$

(a) Prove that the function $u(r, \varphi) := r^\alpha \sin(\varphi/\omega)$ is harmonic, i.e., $\Delta u = 0$.

(b) Prove that $u \in H^2(\Omega_\omega)$ if and only if $\omega \leq \pi$.

Exercise 15. *(integration by parts)*
Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with C^2 boundary. Prove for all $u \in H^1_0(\Omega) \cap H^2(\Omega)$ that

$$\int_\Omega |D^2 u|^2 \, dx = \int_\Omega |\Delta u|^2 \, dx + \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 \left(\nu \cdot \frac{\partial \tau}{\partial s}\right) ds$$

where ν and τ are the unit normal and tangent vectors and $\partial/\partial s$ denotes the derivative with respect to the arclength.

Exercise 16. *(regularity for the Laplacian on convex domains)*
Let $\Omega \subseteq \mathbb{R}^2$ be a convex bounded domain with C^2 boundary and let $u \in H^1_0(\Omega)$ solve $-\Delta u = f$ for given $f \in L^2(\Omega)$. Prove the regularity estimate

$$\|D^2 u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}.$$

Hint: Use Exercise 11.

Exercise 17. *(projections in Hilbert spaces)*
Let H be a Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\| = (\cdot, \cdot)^{1/2}$ and let $P \in \mathcal{L}(H; H)$ be a nontrivial oblique projection, that is $0 \neq P \neq \mathrm{id}$ and $P \circ P = P$. Prove that

$$\|P\|_{\mathcal{L}(H;H)} = \|1 - P\|_{\mathcal{L}(H;H)}.$$