Exercise sheet 3 - Sample solution for group exercises

G 1. (Weak derivative)

Let $I = (a, b)$ be an open interval and let $L_1(I)$ be the space of functions $f : I \to \mathbb{R}$ such that $\int_a^b |f(x)|dx < \infty$. Further denote by $C_c^\infty(I)$ the set of all infinitely differentiable functions f with compact support $\operatorname{supp}(f) = \overline{\{x \in I : f(x) \neq 0\}}$. For $m \in \mathbb{N}$, a function $u \in L_1(I)$ is said to have a mth weak derivative $v \in L_1(I)$ if

$$\int_I u(x)\varphi^{(m)}(x)\mathrm{d}x = (-1)^m \int_I v(x)\varphi(x)\mathrm{d}x$$

for all $\varphi \in C_c^\infty(I)$. Analogously, the weak derivatives are defined for complex-valued, integrable functions.

a) Let $u \in C^1(I) \cap L^1(I)$. Show that the classical and the weak derivative of u coincide in $L_1(I)$. You may use the fact that, for $f \in L^1(I)$, $\int_I f(x)\zeta(x)\mathrm{d}x = 0$ for all $\zeta \in C_c^\infty(I)$ implies $f = 0$.

b) Assume that both f and the weak derivative $f^{(m)}$ are contained in $L_2(\mathbb{T}) \subset L_1(\mathbb{T})$. Show that
$$f^{(m)} = \sum_{k\in\mathbb{Z}} (ik)^m \langle f, e_k\rangle e_k.$$

Solution.

a) Partial integration gives
$$\int_I u'(x)\varphi(x)dx = [u(x)\varphi(x)]_a^b - \int_I u(x)\varphi(x)'dx = -\int_I u(x)\varphi(x)'dx$$
for every test function $\varphi \in C_c^\infty(I)$. Now assume there is another $v \in L_1(I)$ such that $\int_I v(x)\varphi(x)dx = -\int_I u(x)\varphi(x)'dx$ for every test function φ. But then
$$\int (u'(x) - v(x))\varphi(x)dx = 0$$
for every test function and consequently, $u' = v$ in $L_1(I)$.

b) Let $\varphi \in C_c^\infty((-\pi, \pi))$ be a test function. For every Fourier monomial, we have $e_k^{(m)}(x) = (ik)^m e_k(x)$. Since $f \in L_2(\mathbb{T})$, we have $f = \sum_{k\in\mathbb{Z}} \alpha_k e_k$ such that $\sum_{k\in\mathbb{Z}} \alpha_k^2 < \infty$. Likewise, we have $f^{(m)} = \sum_{k\in\mathbb{Z}} \beta_k e_k$ with $\sum_{k\in\mathbb{Z}} \beta_k^2 < \infty$. Now, we calculate
$$\sum_{k\in\mathbb{Z}} \beta_k\langle e_k, \varphi\rangle = \langle f^{(m)}, \varphi\rangle = (-1)^m\langle f, \varphi^{(m)}\rangle = (-1)^m \sum_{k\in\mathbb{Z}} \alpha_k\langle e_k, \varphi^{(m)}\rangle = \sum_{k\in\mathbb{Z}} (ik)^m \alpha_k\langle e_k, \varphi\rangle.$$
Hence, $\langle \sum_{k\in\mathbb{Z}} \beta_k e_k - \sum_{k\in\mathbb{Z}} (ik)^m \alpha_k e_k, \varphi\rangle = 0$. Since φ was arbitrary, the same argument as in a) yields $\sum_{k\in\mathbb{Z}} \beta_k e_k = \sum_{k\in\mathbb{Z}} (ik)^m \alpha_k e_k$. Comparing coefficients yields the result.

□

G 2. (An infinite-dimensional reproducing kernel space)

For $\mathbb{T} = [0, 2\pi]$ being the torus, consider the space $L_2(\mathbb{T})$ of all square-integrable functions $f : [0, 2\pi] \to \mathbb{C}$, equipped with the inner product $\langle f, g\rangle = (2\pi)^{-1} \int_0^{2\pi} f(x)\overline{g(x)}dx$. Let $(e_k)_{k\in\mathbb{Z}}$ be the Fourier ONB in $L_2(\mathbb{T})$. For $m \in \mathbb{N}$, consider the Hilbert space

$$W_0^m(\mathbb{T}) = \left\{ f \in L_2(\mathbb{T}) : \langle f, e_0\rangle = 0 \text{ and } \sum_{k\in\mathbb{Z}\setminus\{0\}} |\langle f, e_k\rangle|^2 k^{2m} < \infty \right\}$$

with inner product $\langle f, g\rangle_{W_0^m} := \sum_{k\in\mathbb{Z}\setminus\{0\}} \langle f, e_k\rangle\langle e_k, g\rangle k^{2m}$.

a) Recall from Sheet 2, H1 c) the Banach space $C(\mathbb{T})$ of continuous, 2π-periodic functions equipped with the supremum norm $\|f\|_\infty = \sup_{x\in\mathbb{T}} |f(x)|$. Show that, for any $f \in W_0^m(\mathbb{T})$, the Fourier series $\sum_{k\in\mathbb{Z}\setminus\{0\}} \langle f, e_k\rangle e_k$ converges pointwise and conclude that $W_0^m(\mathbb{T}) \subset C(\mathbb{T})$.

b) Show that $W_0^m(\mathbb{T})$ is a reproducing kernel Hilbert space with kernel

$$R^1(x, y) = 2 \sum_{k\in\mathbb{Z}\setminus\{0\}} k^{-2m} \cos(k(x - y)).$$

Hint: Recall Exercise G2 and G3 from Sheet 2.

Solution.

a) Consider the sequence of uniformly continuous functions $(g_k)_{k\in\mathbb{Z}\setminus\{0\}}$ given by $g_k(x) = \langle f, e_k\rangle e_k(x)$. By Cauchy-Schwarz on the sequence space, we have

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \|g_k\|_\infty = \sum_{k\in\mathbb{Z}\setminus\{0\}} k^{-m}k^m|\langle f, e_k\rangle| \le \sqrt{\sum_{k\in\mathbb{Z}\setminus\{0\}} k^{-2m}} \sqrt{\sum_{k\in\mathbb{Z}\setminus\{0\}} |\langle f, e_k\rangle|^2 k^{2m}} < \infty$$

Hence, the sequence $(g_k)_k$ converges absolutely which implies that the series $\sum_{k\in\mathbb{Z}\setminus\{0\}} g_k$ converges in C w.r.t. the uniform norm $\|\cdot\|_\infty$. Moreover, the limit of a sequence of uniformly continuous functions is again uniformly continuous.

b) Consider the partial sum $s_n(f) := \sum_{k\neq 0, |k|\le n} \langle f, e_k\rangle e_k$. We have

$$s_n(f)(x) = \sum_{k\neq 0 |k|\le n} \langle f, k^{-2m} e_k(-x)e_k\rangle_{W_0^m} = \langle f, \sum_{k\neq 0, |k|\le n} k^{-2m} e_k(-x)e_k\rangle_{W_0^m}.$$

Let us write $v_x^n = \sum_{k\neq 0, |k|\le n} k^{-2m} e_k(-x)e_k$. The sequence $(v_x^n)_{n\in\mathbb{N}}$ converges in W_0^m. Namely, for $|k| \le n$, $\langle v_x^n, e_k\rangle = k^{-2m} e_k(-x)$ and

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} |k^{-2m} e_k(-x)|^2 k^{2m} = \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{-2m} < \infty.$$

The last estimate holds true since a hyperharmonic series $\sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{-\alpha}$ converges for $\alpha > 1$ according to the Cauchy condensation test. It is now easy to verify that

$$R^1(x, y) := \lim_{n\to\infty} v_x^n(y) = 2 \sum_{k=1}^{\infty} k^{-2m} \cos(k(y - x))$$

has the reproducing property in $W_0^m(\mathbb{T})$.

□