Wissenschaftliches Rechnen II/Scientific Computing II

Sommersemester 2016
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Exercise sheet 3

To be handed in on **Friday, 06.05.2016**

1 Group exercises

G 1. (Weak derivative)

Let $I = (a, b)$ be an open interval and let $L_1(I)$ be the space of functions $f : I \to \mathbb{R}$ such that $\int_a^b |f(x)|dx < \infty$. Further denote by $C_c^\infty(I)$ the set of all infinitely differentiable functions f with compact support $\text{supp}(f) = \overline{\{x \in I : f(x) \neq 0\}}$. For $m \in \mathbb{N}$, a function $u \in L_1(I)$ is said to have a mth weak derivative $v \in L_1(I)$ if

$$\int_I u(x)\varphi^{(m)}(x)\mathrm{d}x = (-1)^m \int_I v(x)\varphi(x)\mathrm{d}x$$

for all $\varphi \in C_c^\infty(I)$. Analogously, the weak derivatives are defined for complex-valued, integrable functions.

a) Let $u \in C^1(I) \cap L^1(I)$. Show that the classical and the weak derivative of u coincide in $L_1(I)$. You may use the fact that, for $f \in L^1(I)$, $\int_I f(x)\zeta(x)\mathrm{d}x = 0$ for all $\zeta \in C_c^\infty(I)$ implies $f = 0$.

b) Assume that both f and the weak derivative $f^{(m)}$ are contained in $L_2(\mathbb{T}) \subset L_1(\mathbb{T})$. Show that
$$f^{(m)} = \sum_{k \in \mathbb{Z}} (ik)^m \langle f, e_k \rangle e_k.$$

G 2. (An infinite-dimensional reproducing kernel space)

For $\mathbb{T} = [0, 2\pi]$ being the torus, consider the space $L_2(\mathbb{T})$ of all square-integrable functions $f : [0, 2\pi] \to \mathbb{C}$, equipped with the inner product $\langle f, g \rangle = (2\pi)^{-1} \int_0^{2\pi} f(x)\overline{g(x)}dx$. Let $(e_k)_{k \in \mathbb{Z}}$ be the Fourier ONB in $L_2(\mathbb{T})$. For $m \in \mathbb{N}$, consider the Hilbert space

$$W_0^m(\mathbb{T}) = \left\{ f \in L_2(\mathbb{T}) : \langle f, e_0 \rangle = 0 \text{ and } \sum_{k \in \mathbb{Z} \setminus \{0\}} |\langle f, e_k \rangle|^2 k^{2m} < \infty \right\}$$

with inner product $\langle f, g \rangle_{W_0^m} := \sum_{k \in \mathbb{Z} \setminus \{0\}} \langle f, e_k \rangle \langle e_k, g \rangle k^{2m}$.

a) Recall from Sheet 2, H1 c) the Banach space $C(\mathbb{T})$ of continuous, 2π-periodic functions equipped with the supremum norm $\|f\|_\infty = \sup_{x \in \mathbb{T}} |f(x)|$. Show that, for any $f \in W_0^m(\mathbb{T})$, the Fourier series $\sum_{k \in \mathbb{Z} \setminus \{0\}} \langle f, e_k \rangle e_k$ converges pointwise and conclude that $W_0^m(\mathbb{T}) \subset C(\mathbb{T})$.

b) Show that $W_0^m(\mathbb{T})$ is a reproducing kernel Hilbert space with kernel
$$R^1(x, y) = 2 \sum_{k=1}^{\infty} k^{-2m} \cos(k(y - x)).$$

Hint: Recall Exercise G2 and G3 from Sheet 2.

2 Homework

H 1. (Weak derivative)

Compute the weak derivative of $h(x) = |x|$. Then compute the weak derivative of $u(x) = \sqrt{|x|}$ and show that $u' \notin L^2([-1,1])$.

(3 Punkte)

H 2. (Bernoulli polynomials)

On the interval $[0,1]$, the *Bernoulli polynomials* are recursively defined via $B_0(x) := 1$,

$$B_n'(x) = nB_{n-1}(x) \quad \text{and} \quad \int\limits_0^1 B_n(x)\,dx = 0 \quad \text{for} \quad n = 1,2,3,\ldots.$$

a) Let $\langle\cdot,\cdot\rangle$ be the inner product given in G2 and $(e_k)_{k\in\mathbb{Z}}$ the Fourier ONB as given in G2. Let $\widetilde{B}_n$ be the transformed version of the Bernoulli polynomial B_n such that the domain is $[0,2\pi]$ instead of $[0,1]$. For $n = 1,\ldots$ and $k \neq 0$, show that

$$\langle \widetilde{B}_{2n}, e_k\rangle = (-1)^{n-1}\frac{(2n)!}{(2\pi k)^{2n}}, \quad \langle \widetilde{B}_{2n+1}, e_k\rangle = i(-1)^n\frac{(2n+1)!}{(2\pi k)^{2n+1}}.$$

b) Show that for $n = 1,2,3,\ldots$ and $x \in [0,1]$, we have:

$$B_{2n}(x) = (-1)^{n-1}2(2n)!\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{(2\pi k)^{2n}} \quad \text{and} \quad B_{2n+1}(x) = (-1)^{n-1}2(2n+1)!\sum_{k=1}^{\infty}\frac{\sin(2\pi kx)}{(2\pi k)^{2n+1}}.$$

Hint: Bear in mind that in general, Fourier series are not defined pointwise.

c) Use the result obtained in a) to rewrite the kernel from Exercise G2 b) in terms of Bernoulli polynomials. To this end, you will need the *fractional part* $\{\cdot\}$, which is given by

$$\{x\} = \begin{cases} x - \lfloor x\rfloor & : x \geq 0, \\ x - \lceil x\rceil & : x < 0. \end{cases}$$

(7 Punkte)

H 3. (Periodic Sobolev space)

Show that the *periodic Sobolev space*

$$W^m(\mathbb{T}) := \left\{ f \in L_2(\mathbb{T}) : \left[(2\pi)^{-1}\int_{-\pi}^{\pi} f(x)dx\right]^2 + (2\pi)^{-1}\int_{-\pi}^{\pi}(f^{(m)}(x))^2dx < \infty \right\}$$

is a reproducing kernel Hilbert space. Specify its kernel.

(6 Punkte)

H 4. (Sobolev space)

Consider the Sobolev space

$$W^1([0,1]) := \{f : [0,1] \to \mathbb{R} : \; f \text{ absolutely continuous }, \; f' \in L_2([0,1])\}$$

equipped with the inner product $\langle f,g\rangle_{W^1} := f(0)g(0) + \int_0^1 f'(x)g'(x)dx$, where f', g' denote weak derivatives. Show that $W^1([0,1])$ is a reproducing kernel Hilbert space with kernel

$$k(x,y) = 1 + \min\{x,y\}.$$

(4 Punkte)