Scientific Computing 2

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Sheet 10

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Exercise 1. (optimal control)

The goal of this exercise is to model a one-dimensional parabolic optimal control problem, discretize it and derive the corresponding Lagrange formulation.
We consider a metal rod and its temperature distribution

$$y\colon [0,1] \times [0,T] \longrightarrow \mathbb{R}$$

with initial condition $y(\cdot,0) = y^0$. Additionally,we assume that we are able to control the heat flux of the metal rod at the end points. More precisely, we model $y(x,t)$ to satisfy the partial differential equation

$$\begin{aligned} y_t - y_{xx} &= f \quad \text{in } [0,1]\times[0,T] \\ -y_x(0,\cdot) &= u_l \quad \text{in } [0,T] \\ y_x(1,\cdot) &= u_r \quad \text{in } [0,T] \\ y(\cdot,0) &= y^0 \quad \text{in } [0,1] \end{aligned}$$

with control parameters $u_l(t)$, $u_r(t)$ and additional enviromental influence $f(x,t)$ (material conditions, additional heat source...). The goal is to influence this temperature distribution such that at time T, it will be close to the desired end state y_d. This should be balanced with respect to the energy needed to advance from y^0 to y_d. A cost functional to this problem can be stated as

$$J(y,u_l,u_r) = \frac{1}{2}\|y(\cdot,T) - y_d\|^2_{L^2[0,1]} + \frac{\alpha}{2}\left(\|u_l\|^2_{L^2[0,T]} + \|u_r\|^2_{L^2[0,T]}\right)$$

which we want to minimize with respect to some constraints on u_l and u_r.
As a first step, we want to do a spatial discretization of the partial differential equation. We interpret $y(x,t) = y(t)(x) = y(t) \in V$ (f likewise), where y is now a function of time mapping into a function space V, which consists of functions defined on $[0,1]$ (for instance $C[0,1]$). The finite-dimensional subspace $V_h \subset V$ with basis $\{\phi_1,\dots,\phi_m\}$ is used to approximate $y(t)$ as

$$y(t) \approx \sum_{i=1}^{m} \underline{y}_i(t)\phi_i$$

with a time-dependent coefficient vector $\underline{y}(t) \in \mathbb{R}^m$.

a) Derive the spatially discretized weak formulation

$$\begin{aligned} M\underline{y}'(t) + K\underline{y}(t) &= L(t)\,, \quad t\in[0,T] \\ M\underline{y}(0) &= I\,. \end{aligned}$$

Here, $M \in \mathbb{R}^{m\times m}$ is the mass matrix with $M_{ij} = \int \phi_i\phi_j$, $K \in \mathbb{R}^{m\times m}$ is the stiffness matrix with $K_{ij} = \int(\phi_i)_x(\phi_j)_x$, $L(t) \in \mathbb{R}^m$ is the load vector with $L_i(t) = \int f(t)\phi_i + \phi_i(0)u_l(t) + \phi_i(1)u_r(t)$, and $I \in \mathbb{R}^m$ are the initial conditions with $I_i = \int y^0\phi_i$.

This is a vector-valued first order ODE with matrix coefficients. We continue with a time discretization. Introducing the time steps $t_n = nT/N$ for $n = 0, \ldots, N$ with spacing $\tau = T/N$ we define $\underline{y}^n = \underline{y}(t_n)$, $L^n = L(t_n)$.

b) Using the implicit Euler scheme, derive the space-time discretized formulation

$$\begin{aligned} (M + \tau K)\, \underline{y}^n &= M\underline{y}^{n-1} + \tau L^n\,, \quad n = 1, \ldots, N \\ M\underline{y}^0 &= I\,. \end{aligned}$$

State a block-matrix formulation that expresses $Y = [\underline{y}^n]_{n=1}^N \in (\mathbb{R}^m)^N$ as the solution of a linear system

$$AY = B\,.$$

c) State the discrete optimization problem using an appropiate discrete cost functional and justify your choice in a few words. Introduce the Lagrangian formalism for this problem using discrete Lagrangian multipliers $\underline{p}^n$ for $n = 1, \ldots, N$ (without restrictions for the control parameters).

d) State the KKT-conditions for the discrete optimization problem, with emphasis on the adjoint equations. Do these equations resemble a certain differential equation?

(20 points)