Scientific Computing 2

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Sheet 3

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Exercise 1. (KKT conditions)

Consider the optimization problem

$$\min_{x \in \mathbb{R}^2} f(x_1, x_2) = (x_1 - 1)^2 + (x_2 - 2)^2$$

with linear inequality constraints

$$\begin{aligned} g_1(x) &= x_1 + x_2 - 1 \leq 0 \\ g_2(x) &= -x_1 \leq 0 \\ g_3(x) &= -x_2 \leq 0 \,. \end{aligned}$$

a) Show that every point in the feasible set is regular and find all points satisfying the KKT conditions. Afterwards, determine the solution x^* to the optimization problem and prove that it is unique.

b) We replace the first inequality constraint with $g_1(x) = (x_1 + x_2 - 1)^3 \leq 0$ (which leaves the feasible set unaffected). Show that x^* does not satisfy the KKT conditions anymore.

(6 points)

Exercise 2. (projection 1)

Let $C \subset \mathbb{R}^n$ be a convex, closed, nonempty set and $y \in \mathbb{R}^n$. Show that the optimization problem

$$\min_{x \in c} \|x - y\|_2^2$$

has a unique solution $P_C(y)$, and that $P_C : \mathbb{R}^n \to C$ is continuous with respect to euclidean distance. Furthermore, use the variational inequality to show the identity

$$(P_C(y_1) - P_C(y_2))^\top (y_1 - y_2) \geq \|P_C(y_1) - P_C(y_2)\|_2^2 \,.$$

(4 points)

Exercise 3. (projection 2)

Let $A \in \mathbb{R}^{m \times n}$ with $m < n$ be a matrix with full rank and $y \in \mathbb{R}^n$. Show that the optimization problem

$$\min_{x \in \mathbb{R}^n} \frac{1}{2} \|x - y\|_2^2$$

with constraint

$$Ax = 0$$

has the solution $x^* = (I - A^\top (AA^\top)^{-1} A) y$.

(4 points)

Exercise 4. (arithmetic and geometric mean)

Consider the optimization problem

$$\min_{x \in \mathbb{R}^n} \sum_{i=1}^{n} x_i$$

with constraints

$$\begin{aligned} \prod_{i=1}^{n} x_i &= c \\ -x_i &\leq 0, \quad i = 1, \ldots, n \end{aligned}$$

for some $c > 0$.

a) First, show that the global solution $x^* \in \mathbb{R}^n$ satisfies $x^* > 0$ (componentwise) and that x^* satisfies the KKT conditions. Afterwards, use this information to compute x^*.

b) Use a) to show the inequality

$$\left(\prod_{i=1}^{n} x_i \right)^{1/n} \leq \frac{1}{n} \sum_{j=1}^{n} x_j$$

for all $x \in (\mathbb{R}_{\geq 0})^n$.

(6 points)