V4E2 - Numerical Simulation

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Exercise sheet 6.

To be handed in on **Tuesday, 05.06.2018**.

Definition 1 (Invariance by translation)**.** Let's suppose to have an uniform infinite space grid. A scheme S is said to be *invariant by translation* (or simply *invariant*) if, defined the translation operator Θ_i such that

$$(\Theta_i V)_j = V_{j+e_i},$$

we have for any $i = 1, \ldots, d$

$$S(\Delta, \Theta_i V) = \Theta_i S(\Delta, V)$$

Exercise 1. (A condition for Lipschitz stability)

We use the notation

$$D_{i,j}[V] = \frac{v_{j+e_i} - v_j}{\Delta x_i}.$$

Prove that a monotone, conserving constants, invariant scheme satisfies the inequality:

$$\frac{\|D_i[S(V)]\|_\infty}{\|D_i[V]\|_\infty} \leq 1$$

This condition is commonly referred as Lipschitz stability.

(3 Punkte)

From now on we consider the linear 1-dimensional advection equation with constant coefficient:

$$u_t(x,t) + cu_x(x,t) = 0, \quad (x,t) \in \mathbb{R} \times (0,T]$$

$$u(x,0) = u_0(x)$$

Exercise 2. (The forward in space and time scheme)

Consider the forward in time, forward in space scheme:

$$0 = \frac{v_i^{j+1} - v_i^j}{\Delta t} + c\frac{v_{i+1}^j - v_i^j}{\Delta x}$$

We set $c = 1$ and initial conditions:

$$u_0(x) = \begin{cases} 1 & -1 \leq x \leq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

Consequently, we use as initial condition for S_Δ

$$v_i^0 = \begin{cases} 1 & -1 \leq i\Delta x \leq 0, \\ 0 & \text{elsewhere.} \end{cases}$$

What can you observe concering convergence of the scheme for the equation stated above? (Hint: an analytic solution is provided by $u(x,t) = u_0(x - ct)$)

(4 Punkte)

Exercise 3. (Numerical experiments - very important!)

Assume $u_0 \in C^1(\mathbb{R})$. First, find a solution to the advection equation by using the methods of characteristics. In this example, we condider a compact space domain $\Omega = [0, 10]$ (instead of the full $\mathbb{R}$) and boundary conditions $u(0, t) = u_0(-ct)$, $u(10, t) = u_0(10 - ct)$.
Then, implement (using a computer and an environment of your choice, for example Python/-Numpy):

- The forward in time, forward in space scheme (the one in the previous exercise):

$$0 = \frac{v_i^{j+1} - v_i^j}{\Delta t} + c\frac{v_{i+1}^j - v_i^j}{\Delta x}.$$

- the forward in time - centered in space (FTCS) scheme:

$$v_i^{j+1} = v_i^j - \frac{c\Delta t}{2\Delta x}(v_{i+1}^j - v_{i-1}^j),$$

- the Upwind scheme:

$$v_i^{j+1} = v_i^j - \frac{c\Delta t}{\Delta x}(v_i^j - v_{i-1}^j)$$

- (if $c > 0$) the Lax-Scheme:

$$v_i^{j+1} = \frac{1}{2}(v_{i+1}^j + v_{i-1}^j) - \frac{c\Delta t}{2\Delta x}(v_{i+1}^j - v_{i-1}^j)$$

for $0 < i < 10$, and the bondary conditions as above in the remaining cases.
We choose parameters:

- $\Omega = [0, 10]$
- $u_0(x) = e^{-10(x-2)^2}$
- $\Delta x = \Delta t = 0.05$
- $c = 0.5$

Plot the approximated solutions for time $t = 0,\ 10,\ 50,\ 100,\ 150$. Play with the parameters Δx, Δt and the velocity c such that the so called CFL condition is not fulfilled anymore, i.e. $\frac{c\Delta t}{\Delta x} > 1$. Is there something interesting to observe comparing the plots?
Hint: also when the CFL condition is satisfied, when the time increases...something unexpected might happen...try then to repeat the approximation, but with a finer meshsize...what do you observe?

(9 Punkte)