# Scientific Computing II

Sommersemester 2019
Prof. Dr. Carsten Burstedde
Biagio Paparella

UNIVERSITÄT BONN

## Exercise Sheet 11.

Due date: **02.07.2019**.

**Exercise 1.** (An inverted version of Cea's lemma) (5 Points)

Suppose that for every $f \in V'$, the solution of

$$a(u, v) = \langle f, v \rangle \text{ for all } v \in V \tag{1}$$

satisfies

$$\lim_{h \to 0} u_h = u \doteq L^{-1} f \tag{2}$$

(where the dot means that we define u to be such an element).
Then we have:

$$\inf_{h>0} \inf_{u_h \in U_h} \sup_{v_h \in V_h} \frac{a(u_h, v_h)}{\|u_h\|_U \|v_h\|_V} > 0 \tag{3}$$

*Hint: this can be seen as a converse of theorem* 3.7 *(Brass, or 2.8 for us). Use (3.10 Braess / 2.2.15) and apply the principle of uniform boundedness.*

**Exercise 2.** (Fredholm Alternative) (5 Points)

Let $H$ be a Hilbert space. Assume that the linear mapping $A : H \to H'$ can be decomposed in the form $A = A_0 + K$, where $A_0$ is $H-$elliptic and $K$ is compact[1]. Show that either $A$ satisfies the inf-sup condition (i.e. hypothesis ii of the isomorphism Theorem 3.6/2.7), or there exists an element $x \in H$, $x \neq 0$, with $Ax = 0$.

**Exercise 3.** (An useful inequality) (5 Points)

In reference to the Saddle Point Problems theory, prove the following elementary but useful inequality: if $a$, $b$, and $c$ positive reals, then $a \leq b + c$ implies $a \leq b^2/a + 2c$.

**Exercise 4.** (inf-sup: space decomposition) (5 Points)

Show that the inf-sup condition (.12 in theorem 2.11, or (4.8) in lemma 4.2 from Braess) is equivalent to the following *decomposition property*: for every $u \in X$ there exists a pair of elements

$$u = v + w \tag{4}$$

with $v \in V$ and $w \in V^{\perp}$ such that

$$\|w\|_X \leq \beta^{-1} \|Bu\|_{M'} \tag{5}$$

where $\beta > 0$ is a constant independent of $u$.

[1] $K : V \to W$ linear operator between Hilbert spaces is said to be *compact*, if for each bounded sequence $\{v_n\} \subseteq V$, the image $\{Kv_n\} \subseteq W$ admits a converging subsequence in $W$.