Formelsammlung

Orthogonalpolynome

a	b	$\omega(x)$	Name	Formel
-1	1	$(1-x)^\alpha(1+x)^\beta$	Jacobi-P.	$P_n^{(\alpha,\beta)}(x) = \frac{1}{2^n n!\omega(x)} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(\omega(x)(x^2-1)^n\right)$
0	∞	e^{-x}	Laguerre-P.	$\ell_n(x) = \frac{1}{2^n n! e^{-x}} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x}(x^2-1)^n\right)$
$-\infty$	∞	e^{-x^2}	Hermite-P.	$H_n(x) = \frac{1}{2^n n! e^{-x^2}} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(e^{-x^2}(x^2-1)^n\right)$

Eigenschaften der Jacobi-Polynome

Orthogonalität:

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x) = \delta_{mn}\frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}$$

3-Term-Rekursion:

$$\begin{aligned} 2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)P_{n+1}^{(\alpha,\beta)}(x) &= (2n+\alpha+\beta+1)[(\alpha^2-\beta^2)+(2n+\alpha+\beta)(2n+\alpha+\beta+2)x]P_n^{(\alpha,\beta)}(x) \\ &= -2(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)P_{n-1}^{(\alpha,\beta)}(x) \end{aligned}$$

Nullstellen: Eigenwerte der tridiagonalen Matrix

$$\mathcal{T}_n = \begin{bmatrix} a_0 & b_1 & 0 & \dots & 0 \\ b_1 & a_1 & b_2 & 0 & \dots \\ 0 & b_2 & a_2 & b_3 & \\ \vdots & & & \ddots & b_{n-1} \\ 0 & \dots & 0 & b_{n-1} & a_{n-1} \end{bmatrix} \in \mathbb{R}^{n,n}$$

mit

$$\begin{aligned} a_j &= \frac{(\beta-\alpha)(\beta+\alpha)}{(2j+\alpha+\beta)(2j+2+\alpha+\beta)}, \\ b_j &= \sqrt{\frac{4j(j+\alpha)(j+\beta)(j+\alpha+\beta)}{(2j-1+\alpha+\beta)(2j+\alpha+\beta)^2(2j+1+\alpha+\beta)}}. \end{aligned}$$

Speziell ist $P_n^{(0,0)}(x) = L_n(x)$ (Legendre-Polynom) und $P_n^{-1/2,-1/2}(x) = T_n(x)$ (Tschebyscheffpolynom).

Moore-Penrose-Inverse

Die Moore-Penrose Inverse $A^\dagger$ von $A \in \mathbb{C}^{m\times n}$ ist die eindeutige Lösung $X \in \mathbb{C}^{n\times m}$ der 4 Gleichungen

$$\begin{aligned} AXA &= A, \\ XAX &= X, \\ (AX)^* &= AX, \\ (XA)^* &= XA. \end{aligned}$$

Orthogonale Transformationen

Householder

Mit

$$v = \begin{cases} \frac{1}{|a_1| \parallel a \parallel_2}(|a_1|a + a_1 \parallel a \parallel_2 e_1) & a_1 \neq 0 \\ a / \parallel a \parallel_2 + e_1 & a_1 = 0 \end{cases}.$$

wird $Pa = \beta e_1$ für $P = I - 2\frac{vv^*}{v^*v}$.

Givens-Rotation

Mit

$$t = \frac{x_2}{|x_1|},\ n = \sqrt{1+|t|^2}, \qquad c = \frac{x_1/|x_1|}{n},\ s = \frac{t}{n} \quad |x_1| \geq |x_2|,$$
$$t = \frac{x_1}{|x_2|},\ n = \sqrt{1+|t|^2}, \qquad s = \frac{x_2/|x_2|}{n},\ c = \frac{t}{n} \quad |x_1| < |x_2|,$$

setzen wir

$$C_{cs} = C_{\theta,\alpha,\beta} = \begin{bmatrix} \overline{c} & \overline{s} \\ -s & c \end{bmatrix} \quad c = e^{\mathrm{i}\alpha}\cos\theta, s = e^{\mathrm{i}\beta}\sin\theta, \quad \alpha,\beta \in [0,2\pi).$$

Dann ist

$$C_{cs}x = \| x \|_2\ e_1.$$

Hutfunktionenbasis auf $[0,1]$

Auf Netz mit den Knoten

$$x_i = ih, \quad h = \frac{1}{n}, i = 0,\dots,n$$

sind

$$\phi_i(x) = \begin{cases} n(x - x_{i-1}) & x \in [x_{i-1}, x_i) \\ n(x_{i+1} - x) & x \in [x_i, x_{i+1}) \\ 0 & \text{sonst} \end{cases} \qquad i = 1,\dots,n-1,$$

PCG-Verfahren

$$\begin{aligned}
x^{(k+1)} &= x^{(k)} + \alpha_k q^{(k)}, \quad \text{mit } \alpha_k = \frac{\langle w^{(k)}, r^{(k)}\rangle}{\langle q^{(k)}, Aq^{(k)}\rangle},\ k \geq 0 \\
r^{(k+1)} &= \begin{cases} r^{(k)} + \alpha_k A q^{(k)} & k \geq 0 \\ Ax^{(0)} - b & k = -1 \end{cases}, , \\
w^{(k+1)} &= C^{-1} r^{(k+1)},\ k \geq 0 \\
q^{(k+1)} &= \begin{cases} \beta_k q^{(k)} - w^{(k+1)} & k \geq 0 \\ w^{(0)} & k = -1 \end{cases}, \quad \text{mit } \beta_k = \frac{\langle w^{(k+1)}, r^{(k+1)}\rangle}{\langle w^{(k)}, r^{(k)}\rangle}
\end{aligned}$$

Arnoldi-Prozeß in Krylowraum

$$\begin{aligned}
v_{j+1} &= Aq_j - \sum_{i=1}^{j} (q_i^* A q_j) q_i \\
q_{j+1} &= \| v_{j+1} \|_2^{-1}\ v_{j+1}
\end{aligned}$$

Als Matrixschreibweise:

$$AQ_{k-1} = Q_k H_{k-1}$$

mit der oberen Hessenbergmatrix

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1,k-1} \\ h_{21} & h_{22} & h_{23} & \dots & h_{2,k-1} \\ 0 & h_{32} & h_{33} & \dots & h_{3,k-1} \\ \dots & \vdots & \vdots & & \\ 0 & 0 & \dots & 0 & h_{k,k-1} \end{bmatrix} \in \mathbb{C}^{k \times (k-1)}, \quad h_{ij} = \begin{cases} q_i^* A q_j & i \leq j \\ v_i^* v_i & i = j+1 \\ 0 & \text{sonst} \end{cases}$$

Gauß-Legendre-Formeln

Anzahl Punkte	Stützstellen	Gewichte
1	$x_1 = 0$	$\omega_1 = 2$
2	$x_{1/2} = \pm\sqrt{1/3}$	$\omega_{1/2} = 1$
3	$x_{1/2} = \pm\sqrt{3/5}$, $x_3 = 0$	$\omega_{1/2} = \frac{5}{9}$, $\omega_3 = \frac{8}{9}$
4	$x_{1/2} = \frac{1}{7} \pm \sqrt{3 - 2\sqrt{6/5}}$, $x_{3/4} = \frac{1}{7} \pm \sqrt{3 + 2\sqrt{6/5}}$	$\omega_{1/2} = \frac{18+\sqrt{30}}{36}$, $\omega_{3/4} = \frac{18-\sqrt{30}}{36}$
5	$x_{1/2} = \frac{1}{3} \pm \sqrt{5 - 2\sqrt{10/7}}$, $x_{3/4} = \frac{1}{3} \pm \sqrt{5 + 2\sqrt{10/7}}$, $x_5 = 0$	$\omega_{1/2} = \frac{322+13\sqrt{70}}{900}$, $\omega_{3/4} = \frac{322-13\sqrt{70}}{900}$, $\omega_5 = \frac{128}{225}$