Klausur zum Modul Ingenieurmathematik II/III **20. August 2013**
für den Bachelorstudiengang Geodäsie und Geoinformation

In der Klausur können insgesamt 75 Punkte erreicht werden.
Zum Bestehen sind mindestens 38 Punkte erforderlich.

Prüfer: Prof. Dr. M. Rumpf, Dr. Martin Lenz

Klausurdauer: 180 Minuten

Bitte Namen, Vornamen und Matrikel-Nummer einsetzen.

Name: ....

Vorname: ....

Matrikel-Nr.: ....

Bitte Schlüsselwort (zur Veröffentlichung der Klausurergebnisse im Netz) eintragen.

Schlüsselwort: ....

Aufgabe	1	2	3	4	5	6	
Punkte	/4	/5	/4	/9	/10	/7	
Aufgabe	7	8	9	10	11	12	$\sum$
Punkte	/4	/5	/6	/7	/7	/7	/75

Note: Viel Erfolg!

Aufgabe 1: Berechnen Sie $\int_0^2 (-x^3 + 2x^2 - 3x + 4)\,\mathrm{d}x$

a) direkt

b) numerisch, mit Hilfe der Kepler'schen Fassregel

$$\int_a^b f(x)\,\mathrm{d}x \approx \frac{b-a}{6}\left(f(a) + 4f(\frac{a+b}{2}) + f(b)\right) .$$

(4 Punkte)

LÖSUNG:

a) (2pts)

$$\begin{aligned}\int_0^1 (-x^3 + 2x^2 - 3x + 4)\,\mathrm{d}x &= \left[-\frac{x^4}{4} + \frac{2x^3}{3} - \frac{3x^2}{2} + 4x\right]_0^2 \\ &= -\frac{16}{4} + \frac{16}{3} - \frac{12}{2} + 8 \\ &= \frac{40}{12} \\ &= \frac{10}{3}\end{aligned}$$

b) (2pts) With $a = 0$, $b = 2$ and $f(x) = -x^3 + 2x^2 - 3x + 4$, we have:

$$\begin{aligned}\int_0^1 (-x^3 + 2x^2 - 3x + 4)\,\mathrm{d}x &\approx \frac{2-0}{6}\Big(f(0) + 4f(1) + f(2)\Big) \\ &= \frac{1}{3}\Big(4 + 4 \cdot 2 + (-2)\Big) \\ &= \frac{10}{3}\end{aligned}$$

Aufgabe 2: Berechnen Sie

a) $\int_0^{\sqrt{\pi}} x \sin(x^2)\, \mathrm{d}x$

b) $\int_0^{\pi} \cos^4 x\, \mathrm{d}x$

(*Tipp*: $\int \cos^2 x\, \mathrm{d}x = \frac{1}{2}\left(x + \sin(x)\cos(x)\right)$)

(5 Punkte)

LÖSUNG:

a) (2pts) Integration by substitution:

$$
\begin{aligned}
&\int_0^{\sqrt{\pi}} \sin(x^2) x\, \mathrm{d}x \\
&= \frac{1}{2}\int_0^{\sqrt{\pi}} \sin(x^2)\, 2x\, \mathrm{d}x = \frac{1}{2}\int_0^{\sqrt{\pi}} \sin(x^2)\, (x^2)'\, \mathrm{d}x = \frac{1}{2}\int_0^{\pi} \sin y\, \mathrm{d}y \\
&\qquad = \frac{1}{2}\left[-\cos y\right]_0^{\pi} = \frac{1}{2}(-\cos\pi + \cos 0) = \frac{-(-1)+1}{2} = 1
\end{aligned}
$$

b) (3pts) Let $I = \int_0^{\pi} \cos^4 x\, \mathrm{d}x$. Then with partial integration:

$$
\begin{aligned}
I &= \int_0^{\pi} \cos^4 x\, \mathrm{d}x \\
&= \int_0^{\pi} (\cos^3 x)(\sin x)'\, \mathrm{d}x \\
&= \left[(\cos^3 x)(\sin x)\right]_0^{\pi} - \int_0^{\pi} (\cos^3 x)' \sin x\, \mathrm{d}x \\
&= 0 - \int_0^{\pi} 3(\cos^2 x)(-\sin x)\sin x\, \mathrm{d}x \\
&= 3\int_0^{\pi} \cos^2 x \sin^2 x\, \mathrm{d}x \\
&= 3\int_0^{\pi} \cos^2 x\, (1 - \cos^2 x)\, \mathrm{d}x \\
&= 3\int_0^{\pi} \cos^2 x\, \mathrm{d}x - 3\int_0^{\pi} \cos^4 x\, \mathrm{d}x \\
&= 3\frac{\pi}{2} - 3I
\end{aligned}
$$

and so $I = \dfrac{3\pi}{2} - 3I \Rightarrow 4I = \dfrac{3\pi}{2} \Rightarrow I = \dfrac{3\pi}{8}$.

Aufgabe 3: a) Zeigen Sie mit Hilfe des Satzes von Taylor, dass

$$\lim_{h\to 0} \frac{-8f(x_0-2h)+5f(x_0+h)+3f(x_0+3h)}{30h} = f'(x_0)$$

gilt, für jede zweimal differenzierbare Funktion $f:\mathbb{R}\to\mathbb{R}$.

b) Zeigen Sie, dass mit der Differenzenquotienten - Formel aus a) Polynome vom Grad 2 exakt differenziert werden.

(4 Punkte)

LÖSUNG:

a) (3pts) Taylor expansions:

$$\begin{aligned} f(x_0-2h) &= f(x_0) - 2hf'(x_0) + 2h^2 f''(\xi_1) \\ f(x_0+h) &= f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(\xi_2) \\ f(x_0+3h) &= f(x_0) + 3hf'(x_0) + \frac{9h^2}{2} f''(\xi_3) \end{aligned}$$

where $\xi_1 \in [x_0-2h, x_0]$, $\xi_2 \in [x_0, x_0+h]$,$\xi_3 \in [x_0, x_0+3h]$. Note that writing $\mathrm{O}(h^2)$ is also correct, but we need the precise error terms for b).

It follows

$$\begin{aligned} -8f(x_0-2h)+5f(x_0+h)+3f(x_0+3h) =& \left(-8f(x_0)+16hf'(x_0)-16h^2 f''(\xi_1)\right) \\ &+\left(5f(x_0)+5hf'(x_0)+\frac{5h^2}{2}f''(\xi_2)\right) \\ &+\left(3f(x_0)+9hf'(x_0)+\frac{27h^2}{2}f''(\xi_3)\right) \\ = 0\cdot f(x_0) + 30hf'(x_0) + h^2 R \end{aligned}$$

where $R = -16f''(\xi_1) + \frac{5}{2}f''(\xi_2) + \frac{27}{2}f''(\xi_3)$. Finally

$$\begin{aligned} &\lim_{h\to 0} \frac{-8f(x_0-2h)+5f(x_0+h)+3f(x_0+3h)}{30h} \\ &\qquad = \lim_{h\to 0} \frac{30hf'(x_0)+h^2R}{30h} = \lim_{h\to 0}\left(f'(x_0)+\frac{R}{30}h\right) = f'(x_0) \end{aligned}$$

b) (1pt) For a second degree polynomial $f(x) = ax^2+bx+c$, $f''(x) = 2a$ for all $x\in\mathbb{R}$. It follows that the residual is

$$R = -16f''(\xi_1) + \frac{5}{2}f''(\xi_2) + \frac{27}{2}f''(\xi_3) = \left(-16+\frac{5}{2}+\frac{27}{2}\right)(2a) = 0$$

and so the formula is exact.

Aufgabe 4: a) Sei $z = a + bi$ eine komplexe Zahl, definieren Sie $\text{Re}(z), \text{Im}(z), |z|$ and $\bar{z}$.

b) Berechnen Sie i^{2013}.

c) Lösen Sie die Gleichung $z^3 = -8$.

d) Berechnen Sie $(3 + 2i)(1 + i)$ und $\dfrac{3 + 2i}{1 - i}$.

(9 Punkte)

LÖSUNG:

a) (2pts)

$$\begin{aligned}
\text{Re}(z) &= a \\
\text{Im}(z) &= b \\
|z| &= \sqrt{a^2 + b^2} \\
\bar{z} &= a - bi
\end{aligned}$$

b) (1pt) $i^{2013} = i^{2012+1} = i^{4\cdot 503+1} = (i^4)^{503} \cdot i^1 = 1^{503} \cdot i = i$

c) (3pts) We write the polar form $z = re^{i\phi}$ and $8 = 8 \cdot (-1) = 8e^{i\pi}$ and so

$$z^3 = -8 \Rightarrow (re^{i\phi})^3 = 8e^{i\pi} \Rightarrow (r^3)e^{3i\phi} = 8e^{i\pi} \Rightarrow \begin{cases} r^3 = 8 \\ \text{ and} \\ 3\phi = 2k\pi + \pi, \quad k \in \mathbb{Z} \end{cases}$$

Since $r \geq 0$, it follows that $r = 2$. For the angle ϕ, there are three different solutions:

- $k = 0 \Rightarrow \phi_0 = \dfrac{\pi}{3}$
- $k = 1 \Rightarrow \phi_1 = \pi$
- $k = 2 \Rightarrow \phi_2 = \dfrac{5\pi}{3}$

(Note: $k = -1, 0, 1$ is also fine.) The three roots are then

$$\begin{aligned}
z_0 &= 2e^{i\phi_0} = 2(\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})) = 2(\frac{1}{2} + i\frac{\sqrt{3}}{2}) = 1 + i\sqrt{3} \\
z_1 &= 2e^{i\phi_1} = 2(\cos\pi + i\sin\pi) = -2 \\
z_3 &= 2e^{i\phi_2} = 2(\cos(\frac{5\pi}{3}) + i\sin(\frac{5\pi}{3})) = 2(\frac{1}{2} - i\frac{\sqrt{3}}{2}) = 1 - i\sqrt{3}
\end{aligned}$$

d) (3pts)

$$(3+2i)(1+i) = 3+2i+3i+2i^2 = (3-2)+(3+2)i = 1+5i$$

$$\frac{3+2i}{1-i} = \frac{(3+2i)(1+i)}{(1-i)(1+i)} = \frac{1+5i}{1^2-i^2} = \frac{1+5i}{2} = \frac{1}{2}+\frac{5}{2}i$$

Aufgabe 5: a) Gegeben eine reelle $n \times n$ Matrix A, definieren Sie die Begriffe *Eigenwert* und *Eigenvektor*.

b) Berechnen Sie die Eigenwerte und Eigenvektoren von

$$A = \begin{pmatrix} 5 & 0 & -2 \\ 0 & 16 & 0 \\ -2 & 0 & 8 \end{pmatrix}.$$

(*Hinweis*: Entwicklen Sie $\det(A - \lambda \mathbb{1})$ nach der zweiten Spalte.)

c) Diagonalisieren Sie die Matrix A.

d) Geben Sie die Länge und Richtung der Hauptachsen des Ellipsoids $5x^2 + 16y^2 + 8z^2 - 4xz = 1$.

(10 Punkte)

LÖSUNG:

a) (2pts) Eine Zahl $\lambda \in \mathbb{R}$ heißt Eigenwert der $n \times n$ Matrix A, wenn es einen Vektor $x \neq 0$ gibt mit $Ax = \lambda x$. Der Vektor x heißt Eigenvektor zum Eigenwert λ.

b) (3pts) Characteristic polynomial of A:

$$\det(A - \lambda I) = (16 - \lambda)\big((5 - \lambda)(8 - \lambda) - (-2)(-2)\big) = (16 - \lambda)\big(\lambda^2 - 13\lambda + 36\big)$$

and so $\det(A - \lambda I) = 0 \Rightarrow \begin{cases} 16 - \lambda = 0 \\ \lambda^2 - 13\lambda + 36 = 0 \end{cases} \Rightarrow \begin{cases} \lambda = 16 \\ \lambda = 9 \\ \lambda = 4 \end{cases}$.

- For $\lambda_1 = 16$, we have $\begin{pmatrix} -11 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & -8 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a = 0 \\ b \in \mathbb{R} \\ c = 0 \end{cases}$ and so the unit length eigenvector is $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

- For $\lambda_2 = 9$, we have $\begin{pmatrix} -4 & 0 & -2 \\ 0 & 7 & 0 \\ -2 & 0 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a \in \mathbb{R} \\ b = 0 \\ c = -2a \end{cases}$ and so the unit length eigenvector is $v_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$ (for $a = \frac{1}{\sqrt{5}}$).

- For $\lambda_3 = 4$, we have $\begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} a = 2c \\ b = 0 \\ c \in \mathbb{R} \end{cases}$ and so the unit length eigenvector is $v_3 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ (for $c = \frac{1}{\sqrt{5}}$).

c) (3pts) The diagonalisation of A is

$$A = UDU^T = \begin{pmatrix} 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 16 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \end{pmatrix}$$

d) (2pts) We note that the ellipsoid has equation $\mathbf{x}^T A \mathbf{x} = 1$, where $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$, and so the principal axes are

- Direction $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ with length $\dfrac{1}{\sqrt{\lambda_1}} = \dfrac{1}{4}$.
- Direction $v_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$ with length $\dfrac{1}{\sqrt{\lambda_2}} = \dfrac{1}{3}$.
- Direction $v_3 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{pmatrix}$ with length $\dfrac{1}{\sqrt{\lambda_3}} = \dfrac{1}{2}$.

Aufgabe 6:

a) Bestimmen Sie die Lösung der Differenzialgleichung

$$\dot{y} = y\cos(2\pi t)$$

mit der Anfangsbedingung $y(0) = 1$.

b) Beschreiben Sie das Eulersche Polygonzugverfahren.

c) Berechnen Sie die ersten zwei Schritte des Eulerschen Polygonzugverfahren für die Differentialgleichung von a) mit $\tau = \frac{1}{2}$.

(7 Punkte)

LÖSUNG:

a) (3pts) Separation of variables:

$$\begin{aligned}\frac{dy}{dt} = y\cos(2\pi t) &\Rightarrow \frac{dy}{y} = \cos(2\pi t)dt\\ &\Rightarrow \int \frac{dy}{y} = \int \cos(2\pi t)dt\\ &\Rightarrow \log y = \frac{\sin(2\pi t)}{2\pi} + C\\ &\Rightarrow y = \exp\left(\frac{\sin(2\pi t)}{2\pi} + C\right)\end{aligned}$$

then $y(0) = 1 \Rightarrow \exp\left(\frac{\sin(0)}{2\pi} + C\right) = 1 \Rightarrow e^C = 1 \Rightarrow C = 0$, and so the solution is

$$y = \exp\left(\frac{\sin(2\pi t)}{2\pi}\right)$$

b) (2pts) If $y'(t) = f(t, y(t))$ and $y_i = y(t_i)$ then $y_{i+1} = y_i + (t_{i+1} - t_i)f(t_i, y_i) \approx y(t_{i+1})$.

c) (2pts) Let $t_0 = 0$ and $y_0 = y(0) = 1$.

i) $t_1 = t_0 + \tau = \frac{1}{2}$ and

$$y_1 = y_0 + \tau f(t_0, y_0) = y_0 + \tau y_0 \cos(2\pi t_0) = 1 + \frac{1}{2} \cdot 1 \cdot \cos(0) = \frac{3}{2}$$

ii) $t_2 = t_1 + \tau = 1$ and

$$y_2 = y_1 + \tau f(t_1, y_1) = y_1 + \tau y_1 \cos(2\pi t_1) = \frac{3}{2} + \frac{1}{2} \cdot \frac{3}{2} \cdot \cos(\pi) = \frac{3}{2} - \frac{3}{4} = \frac{3}{4}$$

Aufgabe 7: a) Geben Sie die Leibniz-formel für die Differenzierung von

$$f(x) = \int_{a(x)}^{b(x)} g(x,y)\,\mathrm{d}y\,.$$

b) Wenden Sie diese Formel auf das Integral

$$f(x) := \int_{x}^{x^2} \frac{\sin(xy)}{y}\,\mathrm{d}y$$

an und geben Sie eine integralfreie Fomulierung von f' an.

(4 Punkte)

LÖSUNG:

a) (2pts)

$$f'(x) = g(x,b(x))b'(x) - g(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial g}{\partial x}(x,y)\,\mathrm{d}y$$

b) (2pts) We have $a(x) = x$, $b(x) = x^2$ and $g(x,y) = \dfrac{\sin(xy)}{y}$ and so

$$\begin{aligned}
f'(x) &= \frac{\sin(x^2\cdot x)}{x^2}\cdot 2x - \frac{\sin(x\cdot x)}{x}\cdot 1 + \int_x^{x^2} \frac{y\cdot\cos(xy)}{y}\,\mathrm{d}y \\
&= \frac{2\sin(x^3)}{x} - \frac{\sin(x^2)}{x} + \int_x^{x^2} \cos(xy)\,\mathrm{d}y \\
&= \frac{2\sin(x^3)}{x} - \frac{\sin(x^2)}{x} + \left[\frac{\sin(xy)}{x}\right]_{y=x}^{y=x^2} \\
&= \frac{2\sin(x^3)}{x} - \frac{\sin(x^2)}{x} + \frac{\sin(x^3)}{x} - \frac{\sin(x^2)}{x} \\
&= \frac{3\sin(x^3)}{x} - \frac{2\sin(x^2)}{x}
\end{aligned}$$

Aufgabe 8: a) Welche Menge wird durch

$$C = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 \leq z^2,\ 0 \leq z \leq 1\}$$

beschrieben?

b) Berechnen Sie des Schwerpunktes

$$\bar{x} = \frac{1}{\text{vol}(C)} \int_C x \, \mathrm{d}x$$

der Menge.

(5 Punkte)

LÖSUNG:

a) (1pt) The set C is a cone aligned with the z-axis and the tip at the origin. The radius of the base is 1 and the height is 1.

b) (3pts) The map (cylindrical coordinates) $g(r, \phi, z) = \begin{pmatrix} r\cos\phi \\ r\sin\phi \\ z \end{pmatrix}$ sends the set

$$P = \{(r, \phi, z) \in \mathbb{R}^3 \,|\, 0 \leq \phi \leq 2\pi, 0 \leq z \leq 1, 0 \leq r \leq z\}$$

to C, i.e. $C = g(P)$.

From the transformation theorem $\text{vol}(C) = \int_C \mathrm{d}\mathbf{x} = \int_P |\det Dg| \, \mathrm{d}\mathbf{y}$. The Jacobian is

$$Dg = \begin{pmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \det Dg = r\cos^2\phi + r\sin^2\phi = r$$

and so

$$\text{vol}(C) = \int_P r \, \mathrm{d}\mathbf{y} = \int_0^{2\pi} \int_0^1 \int_0^z r \, \mathrm{d}r\mathrm{d}z\mathrm{d}\phi = \int_0^{2\pi} \int_0^1 \frac{z^2}{2} \, \mathrm{d}z\mathrm{d}\phi = \int_0^{2\pi} \frac{1}{6} \, \mathrm{d}\phi = \frac{2\pi}{6} = \frac{\pi}{3}$$

For the center of mass we have:

-
$$\begin{aligned} \bar{x} &= \frac{1}{\text{vol}(C)} \int_C x \, \mathrm{d}\mathbf{x} \\ &= \frac{3}{\pi} \int_P r^2 \cos\phi \, \mathrm{d}\mathbf{y} \\ &= \frac{3}{\pi} \int_0^{2\pi} \int_0^1 \int_0^z r^2 \cos\phi \, \mathrm{d}r\mathrm{d}z\mathrm{d}\phi \end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{\pi} \int_0^1 \int_0^z \int_0^{2\pi} r^2 \cos\phi \, \mathrm{d}\phi \mathrm{d}r \mathrm{d}z \\
&= \frac{3}{\pi} \int_0^1 \int_0^z \left[r^2 \sin\phi \right]_0^{2\pi} \, \mathrm{d}r \mathrm{d}z \\
&= 0
\end{aligned}
$$

- $$
\begin{aligned}
\bar{y} &= \frac{1}{\mathrm{vol}(C)} \int_C y \, \mathrm{d}\mathbf{x} \\
&= \frac{3}{\pi} \int_P r^2 \sin\phi \, \mathrm{d}\mathbf{y} \\
&= \frac{3}{\pi} \int_0^1 \int_0^z \int_0^{2\pi} r^2 \sin\phi \, \mathrm{d}\phi \mathrm{d}r \mathrm{d}z \\
&= \frac{3}{\pi} \int_0^1 \int_0^z \left[-r^2 \cos\phi \right]_0^{2\pi} \, \mathrm{d}r \mathrm{d}z \\
&= 0
\end{aligned}
$$

- $$
\begin{aligned}
\bar{z} &= \frac{1}{\mathrm{vol}(C)} \int_C z \, \mathrm{d}\mathbf{x} \\
&= \frac{3}{\pi} \int_P zr \, \mathrm{d}\mathbf{y} \\
&= \frac{3}{\pi} \int_0^1 \int_0^z \int_0^{2\pi} zr \, \mathrm{d}\phi \mathrm{d}r \mathrm{d}z \\
&= \frac{3}{\pi} \int_0^1 \int_0^z 2\pi zr \, \mathrm{d}r \mathrm{d}z \\
&= 6 \int_0^1 \left[\frac{zr^2}{2} \right]_0^z \mathrm{d}z \\
&= 6 \int_0^1 \frac{z^3}{2} \mathrm{d}z \\
&= 6 \left[\frac{z^4}{8} \right]_0^1 \\
&= \frac{6}{8} = \frac{3}{4}
\end{aligned}
$$

It follows that the center of mass is $\bar{\mathbf{x}} = \begin{pmatrix} 0 \\ 0 \\ \frac{3}{4} \end{pmatrix}$.

Aufgabe 9:

a) Zeigen Sie, dass das Produkt von zwei orthogonalen Matrizen orthogonal ist.

b) Zeigen Sie, dass die Spiegelung

$$S = \mathbb{1} - 2nn^T$$

mit $\|n\| = 1$ orthogonal ist.

c) Berechnen sie die QR-Zerlegung der Matrix

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} .$$

(6 Punkte)

LÖSUNG:

a) (2pts) $(AB)(AB)^T = (AB)(B^T A^T) = A(BB^T)A^T = A\mathbb{1}A^T = AA^T = \mathbb{1}$

b) (2pts)

$$\begin{aligned} SS^T &= (\mathbb{1} - 2nn^T)(\mathbb{1} - 2nn^T)^T \\ &= (\mathbb{1} - 2nn^T)(\mathbb{1}^T - 2(n^T)^T n^T) \\ &= (\mathbb{1} - 2nn^T)(\mathbb{1} - 2nn^T) \\ &= \mathbb{1} - 2nn^T - 2nn^T + (2nn^T)(2nn^T) \\ &= \mathbb{1} - 4nn^T + 4nn^T nn^T \\ &= \mathbb{1} - 4nn^T + 4n(n^T n)n^T \\ &= \mathbb{1} - 4nn^T + 4\|n\|^2 nn^T \\ &= \mathbb{1} - 4nn^T + 4nn^T \\ &= \mathbb{1} \end{aligned}$$

so $S^T = S^{-1} \Rightarrow S$ is orthogonal.

c) (2pts)

- $a_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and so $\alpha_1 = -\operatorname{sgn}(a_{11})\|a_1\| = -\sqrt{3^2 + 4^2} = -5$
- $v_1 = a_1 - \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix}$
- $Q^{(1)} = \mathbb{1} - 2\dfrac{v_1 v_1^T}{\|v_1\|^2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \dfrac{2}{8^2 + 4^2} \begin{pmatrix} 8^2 & 8 \cdot 4 \\ 8 \cdot 4 & 4^2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$
- $A^{(1)} = Q^{(1)} A = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} -5 & -\frac{11}{5} \\ 0 & 2 \end{pmatrix}$

The QR-decomposition is then $A = QR$ with $Q = (Q^{(1)})^T$ and $R = A^{(1)}$, i.e.

$$A = \begin{pmatrix} -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} -5 & -\frac{11}{5} \\ 0 & 2 \end{pmatrix}$$

Aufgabe 10: Betrachten Sie die Gleichungen:

$$h(x, y, z) := x^2 + y^2 - 1 = 0$$
$$g(x, y, z) := x - z = 0$$
$$\mathbf{f}(x, y, z) = \begin{pmatrix} h(x, y, z) \\ g(x, y, z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

a) Welche Figuren schneiden sich hier? Was ist die Schnittmenge dieser Figuren? Fertigen Sie eine Skizze der Situation an.

b) Finden Sie einen Punkt P auf der Schnittmenge mit $x = 1$.

c) Berechnen Sie den Gradienten ∇h, ∇g an dem Punkt P und nutzen sie, um eine Tangentenvektor der Schnittmenge zu finden.

(7 Punkte)

LÖSUNG:

a) (3pts) It's an intersection between a cylinder aligned with the z-axis and with radius 1 and a plane with normal $n = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ that goes through the origin. The intersection is an ellipse.

b) (2pts) If $x = 1$ then because $g(x, y, z) = 0 \Rightarrow x - z = 0 \Rightarrow z = x = 1$ and also $h(x, y, z) = 0 \Rightarrow x^2 + y^2 = 1 \Rightarrow y = 0$. So the point is $P = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

c) (2pts) We have $\nabla h = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\nabla g = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ at P. The gradients are normal to the sets $h(x, y, z) = 0$ and $g(x, y, z) = 0$ and therefore to the intersection. It follows that a tangent vector $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ of the intersection needs to be normal to both gradients $v \cdot \nabla h = v \cdot \nabla g = 0$, and so

$$v. \cdot \nabla h = 0 \Rightarrow 2v_1 = 0 \Rightarrow v_1 = 0$$

and

$$v \cdot \nabla g = 0 \Rightarrow v_1 - v_3 = 0 \Rightarrow v_3 = v_1 = 0$$

We conclude that any vector of the form $v = \begin{pmatrix} 0 \\ v_2 \\ 0 \end{pmatrix}$ is tangent to the intersection at P.

Aufgabe 11: Bestimmen Sie die kritischen Punkte der folgenden Funktionen. Sind sie Maxima oder Minima?

a) $f(x) = \exp(\frac{x^3}{3} - x)$

b) $g(x, y) = \exp(\frac{x^3}{3} - x)\exp(\frac{y^3}{3} - y)$

(7 Punkte)

LÖSUNG:

a) (3pts)

$$f'(x) = \exp(\frac{x^3}{3} - x)(\frac{x^3}{3} - x)' = \exp(\frac{x^3}{3} - x)(x^2 - 1)$$

and so the critical points are $x_1 = -1$ and $x_2 = 1$.

$$\begin{aligned} f''(x) &= \left(\exp(\frac{x^3}{3} - x)\right)'(x^2 - 1) + \exp(\frac{x^3}{3} - x)(x^2 - 1)' \\ &= \exp(\frac{x^3}{3} - x)(x^2 - 1)^2 + \exp(\frac{x^3}{3} - x)(2x) \\ &= \exp(\frac{x^3}{3} - x)((x^2 - 1)^2 + 2x) \end{aligned}$$

and then

$$f''(-1) = e^{2/3} \cdot (-2) < 0$$

and so $x_1 = -1$ is a maximum, and

$$f''(1) = e^{-2/3} \cdot 2 > 0$$

and so $x_2 = 1$ is a minimum

b) (4pts) we note that $g(x, y) = f(x)f(y)$ and so

$$\nabla g = \begin{pmatrix} f'(x)f(y) \\ f(x)f'(y) \end{pmatrix}$$

Since $f(x) > 0$ and $f(y) > 0$, the critical points are where $f'(x) = f'(y) = 0$. There are four such points $\{(-1, -1), (1, -1), (-1, 1), (1, 1)\}$. The Hessian is

$$\nabla^2 g = \begin{pmatrix} \partial_{xx} g & \partial_{xy} g \\ \partial_{xy} g & \partial_{yy} g \end{pmatrix} = \begin{pmatrix} f''(x)f(y) & f'(x)f'(y) \\ f'(x)f'(y) & f(x)f''(y) \end{pmatrix}$$

- At $(x, y) = (-1, -1)$,

$$\nabla^2 g(-1,-1) = \begin{pmatrix} f''(-1)f(-1) & 0 \\ 0 & f(-1)f''(-1) \end{pmatrix}$$

  which is negative definite since $f''(-1)f(-1) < 0$. This point is a maximum.
- At $(x, y) = (1, -1)$,

$$\nabla^2 g(1,-1) = \begin{pmatrix} f''(1)f(-1) & 0 \\ 0 & f(1)f''(-1) \end{pmatrix}$$

  which is indefinite since $f''(1)f(-1) > 0$ and $f(1)f''(-1) < 0$. The point is neither minimum nor maximum (it's a saddle point).
- At $(x, y) = (-1, 1)$,

$$\nabla^2 g(-1,1) = \begin{pmatrix} f''(-1)f(1) & 0 \\ 0 & f(-1)f''(1) \end{pmatrix}$$

  which is indefinite since $f''(-1)f(1) < 0$ and $f(-1)f''(1) > 0$. The point is neither minimum nor maximum (it's a saddle point).
- At $(x, y) = (1, 1)$,

$$\nabla^2 g(1,1) = \begin{pmatrix} f''(1)f(1) & 0 \\ 0 & f(1)f''(1) \end{pmatrix}$$

  which is negative definite since $f''(1)f(1) > 0$. This point is a minimum.

Aufgabe 12: Betrachten Sie die Kurve

$$\gamma(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^t \cos(2\pi t) \\ e^t \sin(2\pi t) \end{pmatrix} \in \mathbb{R}^2$$

mit $t \in [0, 2]$.

a) Berechnen Sie die Punkte $\gamma(\frac{i}{4})$, $i = \{0, \ldots, 8\}$, und skizzieren Sie die Kurve.

b) Berechnen Sie die Bogenlänge der Kurve.

c) Bestimmen Sie die Krümmung der Kurve.

(7 Punkte)

LÖSUNG:

a) (2pts)

$$\begin{aligned}
\gamma(0) &= (1, 0) \\
\gamma(\frac{1}{4}) &= (0, e^{1/4}) &&\approx (0, 1.28) \\
\gamma(\frac{1}{2}) &= (-e^{1/2}, 0) &&\approx (-1.65, 0) \\
\gamma(\frac{3}{4}) &= (0, -e^{3/4}) &&\approx (0, -2.12) \\
\gamma(1) &= (e, 0) &&\approx (2.72, 0) \\
\gamma(\frac{5}{4}) &= (0, e^{5/4}) &&\approx (0, 3.5) \\
\gamma(\frac{3}{2}) &= (-e^{3/2}, 0) &&\approx (-4.48, 0) \\
\gamma(\frac{7}{4}) &= (0, -e^{7/4}) &&\approx (0, -5.75) \\
\gamma(2) &= (e^2, 0) &&\approx (7.39, 0)
\end{aligned}$$

The curve is a logarithmic spiral, starting at $(0, e)$ and doing two complete counter-clockwise rotations it ends at $(e^2, 0)$.

b) (2pts) We first calculate the velocity

$$\begin{aligned}
v(t) &= \|\gamma'(t)\| \\
&= \sqrt{x'(t)^2 + y'(t)^2} \\
&= \sqrt{(e^t \cos(2\pi t) - 2\pi e^t \sin(2\pi t))^2 + (e^t \sin(2\pi t) + 2\pi e^t \cos(2\pi t))^2} \\
&= \sqrt{(1 + 4\pi^2) e^{2t}}
\end{aligned}$$

$$= \sqrt{(1+4\pi^2)}\, e^t$$

and so the length is

$$L = \int_0^2 v(t)\,\mathrm{d}t = \sqrt{(1+4\pi^2)} \int_0^2 e^t\,\mathrm{d}t$$
$$= \sqrt{(1+4\pi^2)}(e^2 - e^0) = \sqrt{(1+4\pi^2)}(e^2-1) \approx 40.65$$

c) (3pts) For the curvature, we note that the parametrisation is not arc-length and so we need to use the formula

$$\kappa(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{v(t)^3}$$

The second derivatives are

$$\begin{aligned} x''(t) &= (e^t\cos(2\pi t) - 2\pi e^t \sin(2\pi t))' \\ &= e^t\cos(2\pi t) - 2\pi e^t\sin(2\pi t) - 2\pi(e^t\sin(2\pi t) + 2\pi e^t\cos(2\pi t)) \\ &= (1-4\pi^2)e^t\cos(2\pi t) - 4\pi e^t\sin(2\pi t) \end{aligned}$$

and

$$\begin{aligned} y''(t) &= (e^t\sin(2\pi t) + 2\pi e^t \cos(2\pi t))' \\ &= (e^t\sin(2\pi t) + 2\pi e^t\cos(2\pi t)) + 2\pi(e^t\cos(2\pi t) - 2\pi e^t\sin(2\pi t)) \\ &= (1-4\pi^2)e^t\sin(2\pi t) + 4\pi e^t\cos(2\pi t) \end{aligned}$$

Then

$$\begin{aligned} &x'(t)y''(t) \\ =&(e^t\cos(2\pi t) - 2\pi e^t\sin(2\pi t))\big((1-4\pi^2)e^t\sin(2\pi t) + 4\pi e^t\cos(2\pi t)\big) \\ =&4\pi e^{2t}\cos^2(2\pi t) + (1-12\pi^2)e^{2t}\cos(2\pi t)\sin(2\pi t) - 2\pi(1-4\pi^2)e^{2t}\sin^2(2\pi t) \end{aligned}$$

and

$$\begin{aligned} &y'(t)x''(t) \\ =&(e^t\sin(2\pi t) + 2\pi e^t\cos(2\pi t))\big((1-4\pi^2)e^t\cos(2\pi t) - 4\pi e^t\sin(2\pi t)\big) \\ =&2\pi(1-4\pi^2)e^{2t}\cos^2(2\pi t) + (1-12\pi^2)e^{2t}\cos(2\pi t)\sin(2\pi t) - 4\pi e^{2t}\sin^2(2\pi t) \end{aligned}$$

Subtracting cancels the mixed terms, we have

$$x'(t)y''(t) - y'(t)x''(t) = 2\pi(1+4\pi^2)e^{2t}\cos^2(2\pi t) + 2\pi(1+4\pi^2)e^{2t}\sin^2(2\pi t) = 2\pi(1+4\pi^2)e^{2t}$$

and finally

$$\kappa(t) = \frac{2\pi(1+4\pi^2)e^{2t}}{(\sqrt{1+4\pi^2}\,e^t)^3} = \frac{2\pi}{\sqrt{1+4\pi^2}\,e^t}$$