Institut für Numerische Simulation
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Tutorial Numerical Algorithms
Winter term 2012/2013
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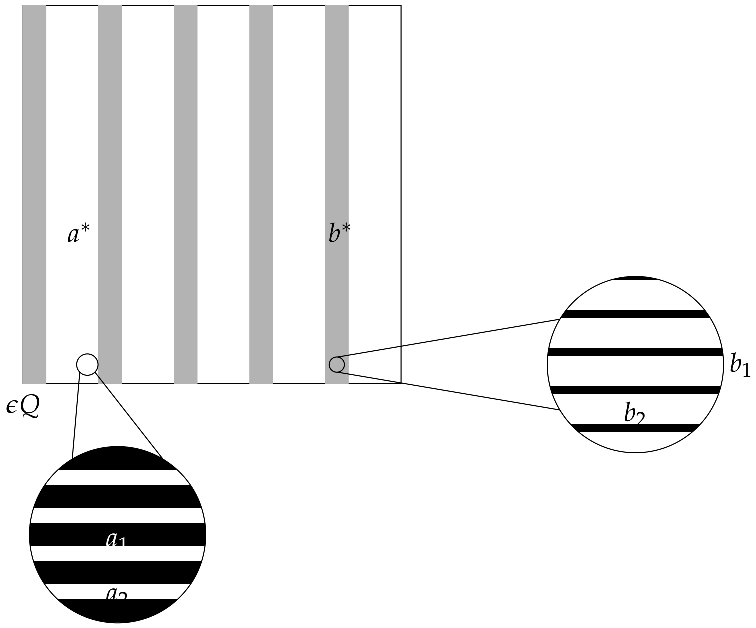
Problem sheet 5 **November 27th, 2012**

Problem 1 (Three-scale material)



Consider a material consisting of a nested periodic micro structure that can be described by a vertical bar pattern on a scale $\epsilon > 0$, where each of these vertical again consists of a micro structure, which can be described by a horizontal bar pattern on a scale $\delta \ll \epsilon$. Let $0 < \lambda < \epsilon$ and let the tensors a^* and b^* represent material properties on the cell ϵQ for $0 \leq x_1 - \lfloor \frac{x_1}{\epsilon} \rfloor \leq \lambda$ and $\lambda < x_1 - \lfloor \frac{x_1}{\epsilon} \rfloor \leq \epsilon$, respecively.

The properties of a^* and b^* are again determined by microstructures, i. e. there are micro cells $\delta Q^{\{a,b\}}$ on ϵQ with bilinear forms $a(.,.)$ and $b(.,.)$ subject to

$$a(y_1, y_2) = a(y_2) = \begin{cases} a_1, & 0 \leq y_2 - \lfloor \frac{y_2}{\delta} \rfloor \leq \mu \\ a_2, & \mu < y_2 - \lfloor \frac{y_2}{\delta} \rfloor \leq \delta \end{cases},$$

$$b(y_1, y_2) = b(y_2) = \begin{cases} b_1, & 0 \leq y_2 - \lfloor \frac{y_2}{\delta} \rfloor \leq \gamma \\ b_2, & \gamma < y_2 - \lfloor \frac{y_2}{\delta} \rfloor \leq \delta \end{cases},$$

where $0 < \mu, \gamma < \delta$.

Determine the effective tensors a^* and b^* and derive a formula for the overall effective tensor c^* corresponding to the bilinear form $c(.,.)$ subject to

$$c(x_1, x_2) = c(x_1) = \begin{cases} a^*(y_1, y_2), & 0 \leq x_1 - \lfloor \frac{x_1}{\epsilon} \rfloor \leq \lambda \\ b^*(y_1, y_2), & \lambda < x_1 - \lfloor \frac{x_1}{\epsilon} \rfloor \leq \epsilon \end{cases} .$$

Problem 2 (Overall computational costs)

Consider the error estimate

$$\|u^{H,h} - u^*\|_{1,2,D} \leq C \left(\mathrm{err}_{HMM} + \left(\frac{h}{\epsilon}\right)^{2l} + H^k \right),$$

where $\mathrm{err}_{HMM} \leq c\epsilon$. Derive the overall computational costs when solving the reconstruction and the H-problem using a CG-minimization.

Problem 3 (Corrector problem for vector fields)

Consider a family of vector fields $u^\epsilon : D \to \mathbb{R}^d$, $D \subset \mathbb{R}^d$, and the optimization problem

$$\min_{u^\epsilon} \int_D W_\epsilon(Du^\epsilon) - f \cdot u_\epsilon \,\mathrm{d}x\,, \quad W_\epsilon(A) = \frac{1}{2} a(x, \frac{x}{\epsilon})\, A : A\,,$$

where $a(.,.)$ is a scalar tensor and $A : B = \mathrm{tr}(A^T B) = \sum_{ij} A_{ij} B_{ij}$. Under appropiate assumptions $u^\epsilon \rightharpoonup u^*$ in $H^{1,2}(D, \mathbb{R}^d)$, where u^* solves

$$u^* = \arg\min_u \int_D W^*(Du) - f \cdot u \,\mathrm{d}x\,, \quad W^*(A) = \frac{1}{2} a^*\, A : A\,.$$

Along the lines of Theorem 2.1, show that there are functions $\chi^{kl} : (x, y) \mapsto \chi^{kl}(x, y)$ that solve the corrector problem

$$\mathrm{div}_y \Big(a(E^{kl} + \nabla_y \chi^{kl}) \Big) = 0\,,$$

where $E^{kl} \in \mathbb{R}^{d,d}$ with $E^{kl}_{ij} = \delta_{ik}\delta_{jl}$ and derive a formula for the effective tensor a^*.