Institut für Numerische Simulation
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Tutorial Numerical Algorithms

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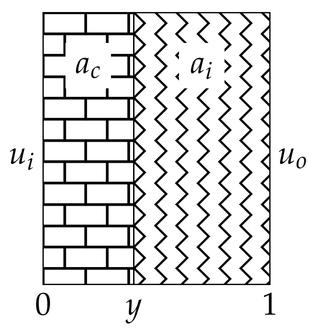
Problem sheet 8 — Januar 8th, 2013

Problem 1 (1D constrained optimization)

In this exercise the temperature profile u along a one dimensional cut through a wall is considered.



The wall is made up of concrete with a high thermal conductivity $a_c > 0$ and low material costs c_c on one side and an insulating material with a low thermal conductivity $0 < a_i < a_c$ but high costs $c_i > c_c$ on the other side. Given inner and outer temperatures u_i and u_o the heat loss shall be minimized while keeping the overall material costs low. Therefore consider the following minimization problem under the constraint that u solves the 1D heat equation with a jumping diffusivity coefficient.

$$\mathbf{J}[y,u] = (c_c y + c_i(1-y)) - (a_i u'(1))$$

Compute the optimal y.

Hint: Determine the unique profile u first.

Problem 2 (Constrained optimization)
Consider the constrained optimization problem:

$$\mathbf{J}[v,u] = vu^2 + v^2 \longrightarrow \min$$
$$\text{s. t.} \quad u = \arg\min_u \, e(v,u) = \arg\min_u \, (u-v)^2 + u^4$$

Write down the Lagrangian $\mathbf{L}$ and derive the Newton method to solve $\nabla \mathbf{L} = 0$.

Problem 3 (Variation of the area functional)
Given a surface M and a (global) parametrization x, i.e. $M = \{x = x(\xi) : \xi \in \omega\}$. Consider a variation in normal direction $\phi(t,x) = x + t\,\vartheta(x)n(x)$. Prove by explicit computation:

$$\left(\int_M \mathrm{d}a\right)_{,M}(\vartheta) := \frac{d}{dt}\int_{\phi(t,M)} \mathrm{d}a\,\Big|_{t=0} = \int_M \kappa\,\vartheta\;\mathrm{d}a\,,$$

where $\kappa = \operatorname{tr} S$ is the mean curvature, $S = g^{-1}h$ the shape operator, $h = Dx^T Dn$.

Hint: Make use of the linearization $\det(\mathbb{1} + \epsilon A) = 1 + \epsilon \operatorname{tr} A + O(\epsilon^2)$.

Problem 4 (Coarea formula)
For a bounded domain $\Omega \subset \mathbb{R}^d$ with polygonal boundary, $\psi : \Omega \to [a,b] \in C^1(\Omega)$, $\nabla\psi \neq 0$ and $g \in H^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ prove the coarea formula:

$$\int_a^b \Big(\int_{[\psi=c]} g\Big)\mathrm{d}c = \int_{[a\le\psi\le b]} g\,|\nabla\psi|\,.$$

(i) Prove the formula for a regular triangulation $\mathcal{T}_h$ of Ω, ψ_h piecewise linear on $T \in \mathcal{T}_h$ and globally continuous and g_h piecewise constant.

(ii) Consider a sequence $\mathcal{T}_h$ with $h \to 0$, ψ_h being the piecewise linear nodal interpolation of ψ and g_h being piecewise constant with $g_h = ⨍_T g$ for each $T \in \mathcal{T}_h$.