Scientific Computing I

Winter semester 2015/2016
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Exercise sheet 1.

Submission due **Tue, 2015-11-03, before lecture**.

Exercise 2. (Modelling with a simple ODE)

A detective finds the body of a murder victim. He would like to determine the time of death from the temperature of the body. He knows the following.

- At *12:36*, the temperature T of the body is measured at 27 degrees.
- According to Newton's law of cooling, the rate of cooling of a body is proportional to the difference of the body's temperature T and the surrounding temperature $R = 20$ degrees.
$$T'(t) \propto T(t) - R$$
- The detective does not know the constants appearing in the model. But, a measurement of the body's temperature at time *13:06* yields 25 degrees.
- At the time of death, the body's temperature is assumed to have been 37 degrees.

Help the detective determine time of death.

a) Choose units for time and determine the corresponding constants of the model.

b) Determine time of death, i.e. t such that $T(t) = 37$, and, convert it into a clock time.

(3 Points)

Exercise 3. (Dimensional Analysis)

In the lectures we have already been introduced to dimensional analysis by considering units of measure. In particular we have already seen nondimensionalization to find simpler and less parameters. It can be also used as a basic check to determine meaningless equations like

$$M = T$$

with M measuring mass in grams, T measuring time. Moreover, let L measure lengths. We call such an equation *dimensionally inconsistent* or *dimensionally non-homogeneous.* In contrast, all meaningful equations are *dimensionally homogeneous.*
However, some quantities have no associated *dimensions* and are called *dimensionless.*

a) Show that angle measures are dimensionless.

b) Show that all trigonometric functions are dimensionless.

c) What is the dimension of ω in $\sin(\omega t)$ with $[t] = T$ being time?

d) Show that the following equation with v being velocity, x position, t time is dimensionally inconsistent

$$v^2 = \frac{x^2}{t} + \frac{x}{t}$$

e) The following are two forms of *Poiseuille's equation.* It describes the flow of fluid through a cylindrical tube with the variables

$$\begin{aligned} &\text{Pressure } P\,, && [P] = ML^{-1}T^{-2}\,,\\ &\text{Tube radius } r\,, && [r] = L\,,\\ &\text{Viscosity of the fluid } \eta\,, && [\eta] = ML^{-1}T^{-1}\,,\\ &\text{Length of the tube } l && [l] = L\,,\\ &\text{Density of the fluid } \rho && [\rho] = ML^{-3} \end{aligned}$$

The two equations for the flow rate f are given by

$$f_1 = \frac{\pi P r^4}{8\eta l}\,, \qquad f_2 = \frac{\pi \rho P r^4}{8\eta l}\,.$$

Determine the dimension/units of these two flow rates and explain the difference.

(4 Points)

Exercise 4. (Nondimensionalization and Redimensionalization)

a) Consider the model for the angle of a nonlinear pendulum

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{g}{l}\sin(\theta)\ .$$

θ and $\sin(\theta)$ are dimensionless already. Hence, for a nondimensionalization, we only need a new scaling for t. Verify that

$$\tau = \sqrt{\frac{g}{l}}t$$

is dimensionless and, compute the corresponding nondimensionalized form of the pendulum equation.

b) Starting with the nondimensionalized form

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2} = -\sin(\theta)$$

of the equation describing a pendulum, compute the redimensionalization, i.e. the differential equation, for

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

using

$$\tau = \sqrt{\frac{g}{l}}t\ .$$

(4 Points)

Exercise 5. (Buckingham π Theorem)
Let $U_j, j = 1, \ldots, k$ be some units (e.g $U_1 = L, U_2 = T$) and $q_i, i = 1, \ldots, n \geq k$ be some variables in those units (e.g. $q_1 = x, [q_1] = L; q_2 = t, [q_2] = T; q_3 = v, [q_3] = \frac{L}{T}$)

$$[q_i] = \prod_{j=1}^{k} U_j^{m_{j,i}} \; .$$

Let $p = n - k$ be the nullity of the dimensional matrix $M \in \mathbb{R}^{k \times n}$

$$M = \left(m_{i,j}\right)_{i,j} = \begin{pmatrix} m_{1,1} & \ldots & m_{1,n} \\ \vdots & \ldots & \vdots \\ m_{k,1} & \ldots & m_{k,n} \end{pmatrix}$$

with columns representing dimensions of variables and k being the rank of the matrix M. Let

$$f(q_1, \ldots, q_n) = 0$$

be a physically meaningful, i.e. dimensionally homogeneous, (differential) equation.
Then, this equation can be restated as

$$F(\pi_1, \ldots, \pi_p) = 0$$

where, for $i = 1, \ldots, p$

$$\pi_i = \prod_{j=1}^{n} q_j^{b_{j,i}} \qquad\qquad M(b_{j,i})_j = 0 \qquad\qquad [\pi_i] = 1$$

is a complete set of dimensionless quantities, i.e. the exponents $(b_{j,i})_j, i = 1, \ldots, p$ form a basis of the kernel $\ker M$.
In the following, assume that the implicit function theorem is applicable and F can be solved for sought variables.

a) Consider the modeling of the period of a pendulum with variables

mass	m	$[m] = M$
length	l	$[l] = L$
gravity	g	$[g] = \frac{L}{T^2}$
angle	θ	$[\theta] = 1$
period	P	$[P] = T$

Show that

$$\pi_1 = \theta \qquad\qquad \pi_2 = \frac{gP^2}{l}$$

form a complete set of dimensionless quantities.

b) Use the Buckingham π theorem and the implicit function theorem to show that

$$P = \sqrt{\frac{l}{g} h(\theta)} \; ,$$

i.e. the period of the pendulum does not depend on the mass but, it does depend on the square root of the length.

(5 Points)