Scientific Computing I

Winter semester 2015/2016
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Exercise sheet 7.

Submission due **Tue, 2015-12-15, before lecture.**

Exercise 22. (The d'Alembert solution for a bounded interval)
Consider the one-dimensional wave equation on a bounded interval $(0, L)$.

$$\begin{aligned} \partial_t^2 u &= c^2 \partial_x^2 u && (x,t) \in (0,L) \times (0,\infty) \\ u(0,t) = u(L,t) &= 0 && t \in [0,\infty) \\ u(x,0) &= u_0(x) && x \in (0,L) \\ \partial_t u(x,0) &= v_0(x) && x \in (0,L) \end{aligned} \tag{1}$$

Consider the $2L-$periodic, odd extensions U_0, V_0 of u_0, v_0, i.e.

$$\begin{aligned} U_0(2kL + x) &= u_0(x) && x \in [0,L], k \in \mathbb{Z} , \\ U_0(2kL - x) &= -u_0(x) && x \in [0,L], k \in \mathbb{Z} , \\ U_0((2k+1)L + x) &= -u_0(L - x) && x \in [0,L], k \in \mathbb{Z} , \\ U_0((2k+1)L - x) &= u_0(L - x) && x \in [0,L], k \in \mathbb{Z} . \end{aligned}$$

Show that the d'Alembert solution from Exercise 20 to initial conditions

$$\begin{aligned} u(x,0) &= U_0 && x \in \mathbb{R} \\ \partial_t v(x,0) &= V_0 && x \in \mathbb{R} \end{aligned}$$

is a solution to (1) as well.

(3 Points)

Exercise 23. (Mean values and harmonic functions)
Let $\Omega \subseteq \mathbb{R}^d$ be open, $u \in C^2(\Omega)$. Show the following.

a) Let u be harmonic in Ω, $\overline{B_R(x)} \subseteq \Omega$ and $r < R$. Then

$$u(x) = \frac{1}{\omega_d r^{d-1}} \int_{\partial B_r(x)} u(y) \mathrm{d}S(y) = \frac{d}{\omega_d r^d} \int_{B_r(x)} u(y) \mathrm{d}y .$$

b) Let

$$u(x) = \frac{1}{\omega_d r^{d-1}} \int_{\partial B_r(x)} u(y) \mathrm{d}S(y)$$

for all $\overline{B_r(x)} \subseteq \Omega$. Then u is harmonic in Ω.

Hint: Consider $\varphi'(r)$ where

$$\varphi(r) = \frac{1}{\omega_d r^{d-1}} \int_{\partial B_r(x)} u(y) \mathrm{d}S(y) .$$

(4 Points)

Exercise 24. (Mollifiers and an application)
Let $\Omega \subseteq \mathbb{R}^d$ be open, $\epsilon > 0$, $\Omega_\epsilon := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\}$.
Consider a so called mollifier

$$\eta \in C^\infty(\mathbb{R}^d)\ , \quad \eta \geq 0\ , \quad \operatorname{supp}\eta := \{x \in \mathbb{R}^d : \eta(x) \neq 0\} \subseteq B_1(0)\ , \quad \int_{B_1(0)} \eta = 1\ ,$$

and set

$$\eta_\epsilon(x) = \frac{1}{\epsilon^d}\eta\Big(\frac{x}{\epsilon}\Big)\ , \qquad \operatorname{supp}\eta_\epsilon \subseteq B_\epsilon(0)\ , \qquad \int_{\mathbb{R}^d} \eta_\epsilon = \int_{B_{\epsilon(0)}} \eta_\epsilon = 1\ .$$

Let

$$f \in L^1_{\text{loc}}(\Omega) := \{f : \Omega \to \mathbb{R} : \forall K \subseteq \Omega, K \text{ compact } : f_K \in L^1(K)\} \supset L^1(\Omega);\ ,$$
$$f_\epsilon(x) := (\eta_\epsilon * f)(x) := \int_{B_\epsilon(x)} \eta_\epsilon(x-y) f(y)\mathrm{d}y\ .$$

a) Show that $f_\epsilon \in C^\infty(\Omega_\epsilon)$.

b) Show that $f_\epsilon \to f$ almost everywhere.

Hint: It holds that

$$\lim_{r \searrow 0} \frac{1}{\omega_d r^{d-1}} \int_{\partial B_r(x)} |f(x) - f(y)|\mathrm{d}y = 0$$

almost everywhere.

c) Let $u \in C(\Omega)$ such that the mean value property

$$u(x) = \frac{1}{\omega_d r^{d-1}} \int_{\partial B_r(x)} u(y)\mathrm{d}y \qquad\qquad \forall \overline{B_r(x)} \subseteq \Omega$$

holds.

Using a), show that $u \in C^\infty(\Omega)$.

(6 Points)

Exercise 25. (Weak derivatives)
Let $\Omega \subset \mathbb{R}^d$ be open and

$$C_c^\infty(\Omega) := \{f \in C^\infty(\Omega) : \operatorname{supp} f \subset \Omega \text{ compact}\}.$$

Let $u, v \in L^1_{\text{loc}}(\Omega)$ and α be some multiindex. Then v is called α-th weak derivative of u iff

$$\forall \varphi \in C_c^\infty : \qquad\qquad \int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v\varphi\ .$$

Show that

a) weak derivates are uniquely determined almost everywhere,

Hint: Consider $(\eta_{\frac{1}{n}} * \chi_A)$ for $A \subset \Omega_\epsilon$.

b) if $u \in C^{|\alpha|}$ then weak and (strong) derivatives agree.

(3 Points)