Exercise sheet 12.

additional exercises for revision, not graded

Exercise 42. (Classification of PDEs)

a) Determine the type of the following PDEs for $u : \mathbb{R}^2 \to \mathbb{R}$ resp. $v : \mathbb{R}^3 \to \mathbb{R}$!

i) $u_{xx} - 2u_{xy} + u_{yy} + u = 0$

ii) $2u_{xx} - u_{xy} + u_{yy} - 3u_x + u_y + 3u = 2xy$

iii) $4u_{xx} - 4u_{xy} - 2u_{yy} + 2u_x = e^x$

iv) $v_{xx} + 2v_{xy} + 2v_{yy} + 4v_{yz} + 5v_{zz} + v_x + v_y = 0$

v) $\mathrm{e}^z v_{xy} - v_{xx} = \log(x^2 + y^2 + z^2)$

b) Determine the areas of the (x, y)-plane in which the PDE

$$(1 + x)u_{xx} + 2xyu_{xy} + y^2 u_{yy} + u_x = 0$$

is elliptic, hyperbolic or parabolic.

Exercise 43. (Fundamental solution of the Laplacian)
Let $a \in \mathbb{R}^d$ with $d \in \{2, 3\}$ and let Δ be the Laplacian.

a) Show that

$$u(x) := \begin{cases} -\frac{1}{2\pi} \log\|x - a\|_2\,, & d = 2 \\ \frac{1}{4\pi\|x-a\|_2}, & d = 3 \end{cases}$$

solves $\Delta u = 0$ in $\mathbb{R}^d \setminus \{a\}$ without (with natural Neumann) boundary conditions.

b) Further show that we have

$$\nabla u(x) = \begin{cases} -\frac{x-a}{2\pi\|x-a\|_2^2}\,, & d = 2, \\ -\frac{x-a}{4\pi\|x-a\|_2^3}\,, & d = 3, \end{cases}$$

for the gradient.

Exercise 44. (One example of a Green's function)

Let
$$G(x,y) := \frac{1}{2}\left((1-x)|y| + x|y-1| - |x-y|\right)$$
be the so called *Green's function* for the interval $(0,1)$.
Show that
$$u(x) := \int_0^1 G(x,y)f(y)\mathrm{d}y + g(x)\ , \qquad\qquad g(x) := g_0 + x(g_1 - g_0)$$
is a solution to the boundary value problem
$$\begin{aligned} -\Delta u(x) &= f(x) & \qquad x &\in (0,1)\ ,\\ u(0) &= g_0\ , \\ u(1) &= g_1\ , \end{aligned}$$
with given $f \in C([0,1])$.

Exercise 45. (Discretization of the Helmholtz-equation)

Consider the Helmholtz-equation
$$-\Delta u(x) + k^2 u(x) = f(x)\ , \qquad\qquad x \in (0,1)$$
with $k \in \mathbb{R}$ and boundary values $u(0) = g_0 \in \mathbb{R}$, $u(1) = g_1 \in \mathbb{R}$.
Compute the resulting linear system that a finite difference discretization using $\Delta \approx \partial^+\partial^-$ and nodes $x_i = ih, i = 1, \ldots, N-1, h = \frac{1}{N}$ yields.

Exercise 46. (Weak derivative)

Consider a piecewise smooth function
$$u(x) = \begin{cases} v(x)\ , & x \in (0,a] \\ w(x)\ , & x \in (a,b) \end{cases} \qquad\qquad v, w \in C^1([0,1])$$
on the interval $(0,1) \ni a$.
Show that $u \in H^1$ if and only if $v(a) = w(a)$.

Exercise 47. (Weak form)

Let Ω be a bounded domain with sufficiently smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Consider the PDE
$$-\nabla \cdot (\kappa \nabla u) = f\ , \qquad\qquad x \in \Omega$$
with mixed boundary values
$$\begin{aligned} u &= g\ , & \qquad x &\in \Gamma_1\ ,\\ \kappa\frac{\partial u}{\partial \nu} &= h\ , & x &\in \Gamma_2\ . \end{aligned}$$

a) Determine a suitable space of test functions for the weak form of this boundary value problem.

b) Derive the weak form of this boundary value problem.

Exercise 48. (Green's second identity)

For sufficiently smooth u, v, prove that
$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial}{\partial\nu}v - v\frac{\partial}{\partial\nu}u\ .$$

Exercise 49. (A pure Neumann problem without compability condition)

Consider the variational form of the pure Neumann problem

$$\begin{aligned} -\nabla \cdot (\kappa \nabla u) &= f \ , & x &\in \Omega \\ \kappa \frac{\partial u}{\partial \nu} &= h \ , & x &\in \partial\Omega \end{aligned}$$

and suppose that

$$\int_\Omega f + \int_{\partial\Omega} h \neq 0 \ .$$

a) Show that

$$u \in H^1(\Omega) \ : \qquad \int_\Omega \kappa \nabla u \cdot \nabla v = \int_\Omega f v + \int_{\partial\Omega} h v \qquad \forall v \in H^1$$

has no solution.

b) Show that with

$$V := \{v \in H^1(\Omega) : \int_\Omega v = 0\}$$

the solution of the variational problem

$$u \in V \ : \qquad \int_\Omega \kappa \nabla u \cdot \nabla v = \int_\Omega f v + \int_{\partial\Omega} h v \qquad \forall v \in V$$

solves the boundary value problem

$$\begin{aligned} -\nabla \cdot (\kappa \nabla u) &= \tilde{f} & x &\in \Omega \\ \kappa \frac{\partial u}{\partial \nu} &= h & x &\in \partial\Omega \end{aligned}$$

where

$$\tilde{f} = f - \frac{\gamma}{|\Omega|} \ , \qquad \gamma = \int_\Omega f + \int_{\partial\Omega} h \ .$$

Exercise 50. (Discontinuous coefficients)

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with smooth boundary. Let Ω be separated into $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ by $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2 = \partial\Omega_1 \cap \partial\Omega_2$.

Let $\alpha_1 \gg \alpha_2 > 0$ and

$$a(x) := \begin{cases} \alpha_1 & x \in \Omega_1 \\ \alpha_2 & x \in \Omega_2 \end{cases}$$

Show that for each classical solution u of the variational problem

$$\int_\Omega a \nabla u \cdot \nabla v \mathrm{d}x = \int_\Omega f v \mathrm{d}x \qquad \forall v \in H_0^1(\Omega)$$

the function $x \mapsto a(x) \nabla u(x) \cdot \nu$ is continuous on Γ (irrespective of the choice of $a(x) = \alpha_i$ on Γ).

Thus, $\nabla u \cdot \nu$ is discontinuous on Γ.

Exercise 51. (Eigenvalue pairs of a stiffness matrix)

Show that the block tri-diagonal matrix

$$A_n = \begin{pmatrix} T & -I_n & & \\ -I_n & \ddots & \ddots & \\ & \ddots & \ddots & -I_n \\ & & -I_n & T \end{pmatrix} \in \mathbb{R}^{n^2 \times n^2} \qquad T = \begin{pmatrix} 4 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{pmatrix} \in \mathbb{R}^{n \times n}$$

with the $n \times n$ identity matrix I_n has eigenvalues

$$\lambda_{k,l} = 4 \sin^2 \left(\frac{k\pi}{2(n+1)} \right) + 4 \sin^2 \left(\frac{l\pi}{2(n+1)} \right)$$

for $k, l = 1, \ldots, n$ and associated eigenvectors

$$(v_{k,l})_{(i-1)n+j} = \sin \left(\frac{ki\pi}{n+1} \right) \sin \left(\frac{\ell j\pi}{n+1} \right)$$

with components $i, j = 1, \ldots, n$.
Hint: addition theorems or using $A_n = \tilde{T} \otimes I_n + I_n \otimes \tilde{T}$ with

$$\tilde{T} := \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \in \mathbb{R}^{n \times n},$$

and the Kronecker-product $\otimes$.