Institut für Numerische Simulation
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Exercises to Wissenschaftliches Rechnen I/Scientific Computing I (V3E1/F4E1)

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Problem sheet 11

Please hand in the solutions on Tuesday January 24!

Exercise 35 **4 Points**

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary and $\mathcal{T}_h$ be any triangulation on Ω. Consider the Raviart-Thomas space $RT_0(\Omega)$ given by

$$RT_0(\Omega) = \Big\{ q \in L^2(\Omega) : \text{for all } T \in \mathcal{T}_h \text{ there exists } a \in \mathbb{R}^2, b \in \mathbb{R} \text{ such that } q(x) = a + bx \text{ for all } x \in T\,, [q]_E \cdot n_E = 0 \text{ for all } E \in \mathcal{E}_h \Big\}\,.$$

Here, $\mathcal{E}_h$ is the set of all edges, n_E denotes the outer normal of T and $[q]_E$ refers to the jump along $E \in \mathcal{E}_h$. Derive an explicit formula for the basis functions of RT_0 using the notation in Figure 1.

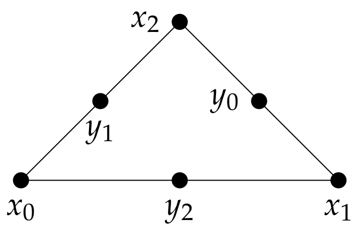

Figure 1: Triangle with vertices and edge midpoints.

Exercise 36 **6 Points**

For $n \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Consider the bilinear forms associated with the Stokes equation

$$a : V \times V \to \mathbb{R}\,, \quad a(u,v) = \int_\Omega \sum_{i=1}^n \nabla u_i \cdot \nabla v_i \, \mathrm{d}x\,,$$
$$b : V \times W \to \mathbb{R}\,, \quad b(v,q) = -\int_\Omega (\mathrm{div} v) \cdot q \, \mathrm{d}x\,,$$

where $V = H_0^1(\Omega, \mathbb{R}^n)$ and $W = \{q \in L^2(\Omega, \mathbb{R}^n) : \int_\Omega q \, \mathrm{d}x = 0\}$. Furthermore, we introduce

$$\tilde{a} : V \times V \to \mathbb{R}\,, \quad \tilde{a}(u,v) = 2 \int_\Omega \sum_{i,j=1}^n e_{ij}(u) e_{ij}(v) \, \mathrm{d}x$$

with $e_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$. Prove that $a(u,v) = \tilde{a}(u,v)$ for all $u, v \in \{f \in V : b(f,q) = 0 \text{ for all } q \in W\}$.

Exercise 37 **4 Points**

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary and $\mathcal{T}_h$ a triangulation on Ω. Let $V_h = RT_0(\Omega, \mathbb{R}^2)$ (see exercise 35) and $W_h \subset \{f \in L^2(\Omega, \mathbb{R}^2) : \int_\Omega f \, \mathrm{d}x = 0\}$ be any finite-dimensional function space. Furthermore, we set

$$b : V_h \times W_h \to \mathbb{R}\,, \quad b(v,w) = -\int_\Omega (\mathrm{div} v) \cdot w \, \mathrm{d}x\,.$$

Show that (see exercise 33 for the definition of $\|\cdot\|_{H(\mathrm{div})}$)

$$\inf_{w \in W_h} \sup_{v \in V_h} \frac{|b(v,w)|}{\|v\|_{H(\mathrm{div})} \|w\|_{L^2(\Omega)}} = 0$$

already implies that there exists $w \in W_h \setminus \{0\}$ such that $b(v,w) = 0$ for all $v \in V_h$.

Exercise 38 **6 Points**

Let $\Omega \subset \mathbb{R}^n$ be a polygonal domain, $\mathcal{T}_h$ be triangulation of Ω, $f \in L^2(\Omega, \mathbb{R}^2)$, and $\mathcal{I}_h$ be the Lagrange interpolation operator w.r.t. the nodes of $\mathcal{T}_h$. To discretize the Stokes equation (see exercise 36), we consider the finite element spaces

$$V_h = \{v_h \in C^0(\Omega, \mathbb{R}^n) \ : \ v_h \in \mathcal{P}_1(T) \ \forall T \in \mathcal{T}_h \, , v_h\big|_{\partial\Omega} = 0\} \, ,$$
$$W_h = \left\{ p_h : \Omega \to \mathbb{R}^n \ : \ p_h \in \mathcal{P}_0(T) \ \forall T \in \mathcal{T}_h \, , \int_\Omega p_h \, \mathrm{d}x = 0 \right\} \, .$$

Then we solve the saddle point problem

$$\begin{aligned} a(u_h, v_h) + b(v_h, p_h) &= (\mathcal{I}_h(f), v_h)_{0,2} && \forall v_h \in V_h \, , \\ b(u_h, q_h) &= 0 && \forall q_h \in W_h \, . \end{aligned}$$

(i.) Construct a basis of W_h.
(ii.) Show that solving the saddle point problem corresponds to solving a linear system

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} F \\ 0 \end{pmatrix} \, . \tag{1}$$

Derive the matrix representation of A and B.
(iii.) Prove that the sequence $(U^{(k)}, P^{(k)})$ obtained with Algorithm 1 converges to the solution (U, P) of (1) if THRESHOLD $= 0$.

Algorithm 1: Saddle point based algorithm.

Data: $P^{(0)}$, $\alpha \in (0, \frac{2}{\|BA^{-1}B^T\|})$, THRESHOLD ≥ 0

Set $k := 1$;
repeat
  Solve $AU^{(k)} := F - B^T P^{(k-1)}$;
  Set $P^{(k)} := P^{(k-1)} + \alpha B U^{(k)}$;
  Set $k := k + 1$;
until $\|P^{(k)} - P^{(k-1)}\| \leq$ THRESHOLD;
