Institut für Numerische Simulation
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Exercises to Wissenschaftliches Rechnen I/Scientific Computing I (V3E1/F4E1)

Winter 2016/17

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Problem sheet 12

Please hand in the solutions on Tuesday January 31!

Exercise 39 **6 Points**

Let Ω be a polygonal domain with regular triangulation $\mathcal{T}_h$. The triangulation $\mathcal{T}_{\frac{h}{2}}$ is obtained by adding new vertices at the midpoints of the edges of each element and joining the vertices correspondingly (see Figure 1). Consider the finite element spaces

$$V_h = \left\{ v_h \in C^0(\overline{\Omega})^2 : v_h|_{\tilde{T}} \in \mathcal{P}_1^2 \text{ for all } \tilde{T} \in \mathcal{T}_{\frac{h}{2}} \text{ and } v_h = 0 \text{ on } \partial\Omega \right\},$$
$$W_h = \left\{ p_h \in C^0(\overline{\Omega}) : p_h|_T \in \mathcal{P}_1 \text{ for all } T \in \mathcal{T}_h \text{ and } \int_\Omega p_h \,\mathrm{d}x = 0 \right\}.$$

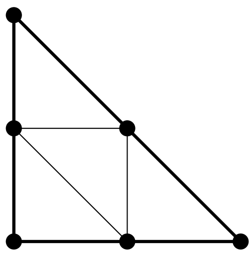

Figure 1: Triangulations $\mathcal{T}_h$ (thick) and $\mathcal{T}_{\frac{h}{2}}$.

(i.) Show that the discrete inf-sup conditon for Stokes problem is satisfied.
Hint: Follow the lines of Lemma 4.14.
(ii.) Let $(v, p) \in H^{2,2}(\Omega) \times H^{1,2}(\Omega)$ be the solution to Stokes problem and (v_h, p_h) the associated discrete solution. Show the estimate

$$\|v - v_h\|_{1,2,\Omega} + \|p - p_h\|_{0,2,\Omega} \leq Ch(\|v\|_{2,2,\Omega} + \|p\|_{1,2,\Omega}).$$

Exercise 40 **4 Points**

Let $\rho > 0$, $n \geq 1$, $p, q \in [1, \infty]$ and $0 \leq m \leq l$ be fixed. Consider a bounded domain $\Omega \subset \mathbb{R}^n$ with polygonal boundary and $\mathcal{T}_h$ a triangulation of Ω such that $\rho h \leq \text{diam}(T) \leq h$ for every $T \in \mathcal{T}_h$. Show that for any $v_h \in V_h \subset H^{l,p}(\Omega) \cap H^{m,q}(\Omega)$, where V_h is a finite element space, and any $T \in \mathcal{T}_h$ the inequality

$$\|v_h\|_{l,p,T} \leq C h^{m-l+\frac{n}{p}-\frac{n}{q}} \|v_h\|_{m,q,T}$$

holds true, where C does neither depend on v_h nor on h.

Exercise 41 **6 Points**

Let $\Omega = [0,1]^2$, $\mathcal{M}_h$ be a regular mesh on Ω composed of quadratic elements and

$$V_h := \left\{ v_h \in L^2(\Omega, \mathbb{R}^2) : v_h\big|_K \in \mathcal{Q}_1^2 \text{ for all } K \in \mathcal{M}_h, \text{ and } v_h = 0 \text{ on } \partial\Omega \right\},$$
$$W_h := \left\{ w_h \in L^2(\Omega, \mathbb{R}) : w_h\big|_K \in \mathcal{Q}_1 \text{ for all } K \in \mathcal{M}_h, \int_\Omega w_h = 0 \right\}.$$

Recall that $\mathcal{Q}_1$ was introduced in exercise 10. Consider the bilinear forms

$$a : V_h \times V_h \to \mathbb{R}, \quad a(u_h, v_h) = \int_\Omega \sum_{i=1}^{2} \nabla(u_h)_i \cdot \nabla(v_h)_i \,\mathrm{d}x,$$
$$b : V_h \times W_h \to \mathbb{R}, \quad b(v_h, q_h) = -\int_\Omega (\mathrm{div} v_h) \cdot q_h \,\mathrm{d}x$$

associated with Stokes problem. Construct a counterexample showing that the inf-sup-condition is not satisfied, i.e.

$$\inf_{q_h \in W_h} \sup_{v_h \in V_h} \frac{|b(v_h, q_h)|}{\|v_h\|_{1,2,\Omega} \|q_h\|_{0,2,\Omega}} = 0.$$