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Exercises to Wissenschaftliches Rechnen I/Scientific Computing I (V3E1/F4E1)

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Problem sheet 5

Please hand in the solutions on Tuesday November 29!

Exercise 14 **4 Points**

For a triangle T we define a refinement

- in red by dividing T into four triangles with new nodes on the edge midpoints,
- in green w.r.t. an edge E by dividing T into two triangles with a new node on the edge midpoint of E.

Figure 1: Left: red-refinement. Right: green-refinement.

Let $\mathcal{T} = \{T_i\}_{i \in I}$ be a triangular mesh consisting of elements T_i. Let $r, g : I \to \{0, 1\}$ be functions, which indicate if T_i is already red resp. green refined. Furthermore, let $\mathrm{mark} : I \to \{0, 1\}$ be a marking function which decides if an element T_i has to be refined. We denote by $\mathrm{nb}(T_i, E)$ the neighboring element of T_i s.t. E is a common edge. Now, a red-green refinement of $\mathcal{T}$ is based on the following assumptions:

- a marked element (i.e. $\mathrm{mark}(i) = 1$) is replaced by a red-refinement,
- after a red-refinement, neighboring elements are replaced by green-refinements,
- if the element, which has to be red-refined, has a neighboring element that is already green-refined, the green-refinement has first to be replaced by a red-refinement.

Think about possible refinement patterns and write a pseudo code algorithm to refine $\mathcal{T}$ with a red-green refinement.

Exercise 15 **4 Points**

Consider the reference domain $\widehat{\omega} = [-1, 1]$,

$$\begin{aligned} &T_1 = [0, 2h], \quad T_2 = [2h, 3h], \quad w_h = T_1 \cup T_2, \\ &\widehat{T}_1 = [-1, 0], \quad \widehat{T}_2 = [0, 1], \quad \widehat{w} = \widehat{T}_1 \cup \widehat{T}_2, \end{aligned}$$

and the affine transformation $F : \widehat{\omega} \to \omega_h$. Let $u(x) = x$, $\widehat{u} = u \circ F$ and $P_{L^2}\widehat{u}$ be the local L^2-projection of $\widehat{u}$ onto $\mathcal{P}_1$, i.e. $\int_{\widehat{\omega}} (P_{L^2}\widehat{u} - \widehat{u}) \cdot 1 \, \mathrm{d}t = \int_{\widehat{\omega}} (P_{L^2}\widehat{u} - \widehat{u}) \cdot t \, \mathrm{d}t = 0$. Show that the local projection error is only of first order, i.e.

$$\frac{\|u - (P_{L^2}\widehat{u}) \circ F^{-1}\|_{0,2,\omega_h}}{\|u\|_{2,2,\omega_h}} \leq \frac{h}{4\sqrt{3}\sqrt{1 + 3h^2}}.$$

Exercise 16 **4 Points**

Let H be a Hilbert space and $V \subset H$ a dense subspace such that $V \hookrightarrow H$ is continuous. Furthermore, let V_h be a subspace of V, $a(\cdot, \cdot)$ and $a_h(\cdot, \cdot)$ are coercive and bounded bilinear forms, and l and l_h are continuous functionals on V and V_h, respectively. We denote by $u \in V$ and $u_h \in V_h$ the solutions of the associated variational problems, i.e. $a(u, v) = l(v)$ for all $v \in V$, $a_h(u_h, v_h) = l_h(v_h)$ for all $v_h \in V_h$, and $\varphi_\eta \in V$ for a $\eta \in H$ is the solution of $a(v, \varphi_\eta) = (\eta, v)_H$ for all $v \in V$. Prove the error estimate (for a constant $C > 0$)

$$\begin{aligned} \|u - u_h\|_H \leq \sup_{\eta \in H} \frac{1}{\|\eta\|_H} \inf_{\varphi_h \in V_h} \big(&C\|u - u_h\|_V \|\varphi_\eta - \varphi_h\|_V \\ &+ |a(u_h, \varphi_h) - a_h(u_h, \varphi_h)| + |l(\varphi_h) - l_h(\varphi_h)| \big). \end{aligned}$$

Exercise 17 **4 Points**

Let $\Omega \subset \mathbb{R}^n$ be a non-empty, bounded and open set. Consider the biharmonic equation $\Delta^2 u = f$ in Ω , $u = u^\partial$ and $\partial_n u = \partial_n u^\partial$ on $\partial\Omega$ for given $f \in L^2(\Omega)$ and $u^\partial \in H^{2,2}(\Omega)$. Here, $\Delta^2 u := \Delta\Delta u$.

(i.) Prove that $u \in C^4(\overline{\Omega})$ is a solution of the above biharmonic equation if and only if u satisfies the boundary conditions and $\int_\Omega \Delta u \, \Delta\varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}x$ $(\forall \varphi \in H_0^{2,2}(\Omega))$ holds true.

(ii.) Show that $\int_\Omega \Delta g \Delta h \, \mathrm{d}x = \int_\Omega \sum_{i,j=1}^n \partial_{i,j}^2 g \partial_{i,j}^2 h \, \mathrm{d}x$ for all $g, h \in H_0^{2,2}(\Omega)$. In particular, $|g|_{2,2,\Omega} = \|\Delta g\|_{0,2,\Omega}$.

(iii.) Let Ω be a convex domain with smooth boundary. Prove that a unique weak solution $u \in H^{2,2}(\Omega)$ of the biharmonic equation in (i.) exists subjected to the boundary conditions $u = u^\partial$ and $\partial_n u = \partial_n u^\partial$ on $\partial\Omega$.

Hint: Consider the bilinear form $a(g, h) = \int_\Omega \Delta g \Delta h \, \mathrm{d}x$ for $g, h \in H_0^{2,2}(\Omega)$. Recall that the Poincaré inequality implies $\|g\|_{1,2,\Omega} \leq \widetilde{C}_P |g|_{2,2,\Omega}$ for a constant $\widetilde{C}_P > 0$.