Institut für Numerische Simulation
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Exercises to Wissenschaftliches Rechnen I/Scientific Computing I (V3E1/F4E1)

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Problem sheet 8

Please hand in the solutions on Tuesday December 20!

Exercise 23 **4 Points**

For $m, n \geq 2$ let $\mathcal{G} = \{1, \ldots, m\} \times \{1, \ldots, n\}$ be a rectangular grid. On $\mathcal{G}$, a regular triangular mesh $\mathcal{T}_h$ can be constructed by using the points in $\mathcal{G}$ as vertices for the triangles as shown in Figure 1 (in this example, $m = 8$ and $n = 4$ and the circles $\circ$ correspond to the elements of $\mathcal{G}$). On the triangular mesh $\mathcal{T}_h$, we consider the

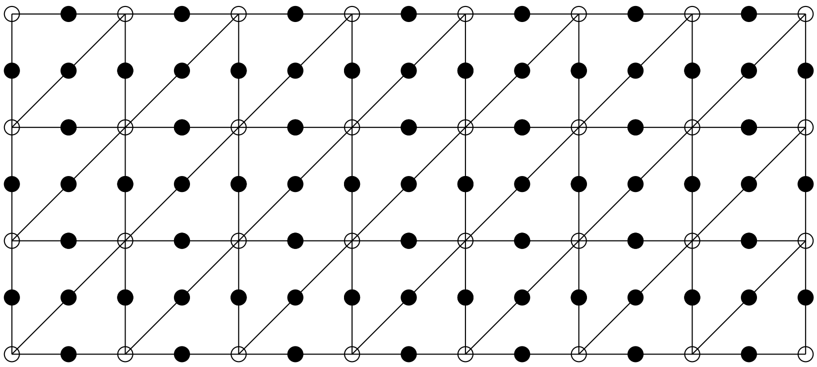
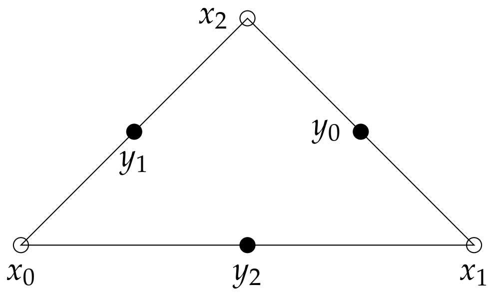


Figure 1: Triangular mesh for Crouzeix-Raviart elements.

Crouzeix-Raviart finite element space $\mathcal{V}_h$ (the degrees of freedom for the Crouzeix-Raviart elements are the filled circles $\bullet$ in Figure 1). Derive an explicit formula for the total number of degrees of freedom of $\mathcal{V}_h$ in terms of m and n. Compare your result with the number of degrees of freedom of the $\mathcal{P}_1$-finite element space on $\mathcal{T}_h$.

Exercise 24 **4 Points**

Consider the following triangle $T \subset \mathbb{R}^2$ with vertexes x_0, x_1 and x_2:



Derive an explicit formula for the local stiffness matrix of the Crouzeix-Raviart finite element on T in terms of x_0, x_1 and x_2, where y_0, y_1 and y_2 represent the degrees of freedom located at the midpoints of the edges.

Exercise 25 **4 Points**

Let $\Omega = (0,1)$, $\mathcal{T}_h$ be a given triangulation on Ω and $f \in L^2(\Omega)$. Furthermore, let $u \in H^1(\Omega)$ be the weak solution of

$$\begin{aligned} -u'' &= f \qquad \text{in } \Omega\,, \\ u(0) = u(1) &= 0\,. \end{aligned}$$

Derive an a posteriori error estimate for the discrete solution $u_h \in \mathcal{V}_h$ w.r.t. the $H^1(\Omega)$-seminorm $\left(\int_\Omega |u'|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}$, where $\mathcal{V}_h$ is the space of $\mathcal{P}_1$-finite elements on $\mathcal{T}_h$.
Hint: Follow the proof of Theorem 2.3 and use Lagrange interpolation.

Exercise 26 **4 Points**

Let $\Omega = (0,1)^2$ and $\mathcal{T}_h$ be a given triangulation on Ω. For $f \in L^2(\Omega)$ consider

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega\,. \end{aligned} \tag{1}$$

We denote by u the weak solution of (1) and by u_h the discrete solution on the space of $\mathcal{P}_1$-finite elements on $\mathcal{T}_h$. Find some $f \in L^2(\Omega)$ such that $u \neq u_h$, but $\eta_T = 0$ for all $T \in \mathcal{T}_h$.