Priv.-Doz. Dr. Christian Rieger
Fabian Hoppe

UNIVERSITÄT BONN

13^{th} exercise sheet

Submission on January 23, before the lecture

Exercise 1.

We consider the problem

$$-(au')' = 1 \quad \text{on } [0,1], \qquad u(0) = u(1) = 1, \tag{1}$$

with $a(x) = \begin{cases} 1 & \text{if } x \leq \frac{1}{2}, \\ 10 & \text{if } x > \frac{1}{2} \end{cases}$. The initial finite element space V_0 is given by piecewise linear polynomials on the partition given by the points $x_k = k/4$, $k = 0, \ldots, 4$.

a) Compute the Galerkin approximation u_0 to (1) with Ansatz space V_0.

b) Compute the residual a posteriori error estimate from the lecture for u_0. Which element of the partition has the largest contribution?

Exercise 2.

Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary. We consider the saddle point problem

$$\begin{aligned} a(w, v) + b(v, u) &= 0 & \forall v \in X = H^1(\Omega) \\ b(w, \varphi) &= \int_\Omega f\varphi \,\mathrm{d}x & \forall \varphi \in M = H_0^1(\Omega) \end{aligned}$$

with the Hilbert spaces $X = H^1(\Omega)$, $M = H_0^1(\Omega)$ and the continuous bilinear forms

$$a(w, v) := \int_\Omega wv \,\mathrm{d}x, \qquad b(w, \varphi) = \int_\Omega \nabla w \nabla \varphi \,\mathrm{d}x.$$

a) Show that the inf-sup condition

$$\sup_{v \in X} \frac{b(v, u)}{\|v\|_X} \geq \beta \|u\|_X \qquad \forall u \in M$$

holds.

b) Show that the saddle point problem has a unique solution $(u, w) \in X \times M$ for any $f \in L^2(\Omega)$.

c) Which PDE (in strong form) is encoded in the saddle point problem?

Exercise 3.

In the last exercise that should demonstrate the benefit of adaptivity (sheet 10, exercise 2) there was a mistake, that made it impossible to observe this benefit. So, here is the corrected version of this exercise:

For some $\alpha \in (0,1]$ we denote by

$$C^{0,\alpha}([0,1]) := \left\{ u\colon [0,1] \longrightarrow \mathbb{R} \middle| \sup_{x,y\in[0,1]} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\}$$

the space of functions on $[0,1]$ being Hölder continuous with exponent α. This spaces equipped with the norm

$$\|u\|_{C^{0,\alpha}} := \|u\|_{C^0} + |u|_{C^{0,\alpha}} := \sup_{x\in[0,1]} |u(x)| + \sup_{x,y\in[0,1]} \frac{|u(x)-u(y)|}{|x-y|^\alpha}.$$

Given $n \in \mathbb{N}$ we introduce two different sets of functions:

$$V_n := \left\{ x \mapsto \sum_{k=0}^{n-1} c_k \mathbf{1}_{[k/n,(k+1)/n)}(x) \middle| c_0, \dots, c_{n-1} \in \mathbb{R} \right\},$$

the space of piecewise constant functions with respect to a *uniform partition of* $[0,1]$ *into* n *subintervals*, and

$$A_n := \left\{ x \mapsto \sum_{k=0}^{n-1} c_k \mathbf{1}_{[x_k,x_{k+1})}(x) \middle| c_0, \dots, c_{n-1} \in \mathbb{R}, 0 = x_0 < x_1 < \dots < x_{n-1} < x_n = 1 \right\},$$

the set (no vector space!) of all functions that are piecewise constant with respect to *an arbitray partition of* $[0,1]$ *into* n *subintervals.*

a) Let $u\colon [0,1] \longrightarrow \mathbb{R}$ be continuous and $\alpha \in (0,1]$. Show that

$$\inf_{v_n\in V_n} \|v_n - u\|_{L^\infty} = \mathcal{O}(n^{-\alpha})$$

holds if and only if $u \in C^{0,\alpha}([0,1])$.

b) Let $u \in W^{1,1}([0,1])$. Show that

$$\inf_{v_n\in A_n} \|v_n - u\|_{L^\infty} = \frac{|u|_{W^{1,1}}}{2} \cdot n^{-1}.$$

Hint: Choose the partition points x_k in such a way that $\int_{x_k}^{x_{k+1}} |u'| \,\mathrm{d}x = \frac{1}{n}|u|_{W^{1,1}}$ for all $k = 0, \dots, n-1$.

c) Compare the two different approximation rates for the functions $u(x) = x^\beta$, $\beta \in (0,1]$. What do you observe?

Programming exercise 1. We consider two different PDEs:

- $\Omega = [0,1]^2 \setminus [0.5,1]^2 \subset \mathbb{R}^2$ we consider the PDE

$$\begin{aligned} -\Delta u(x,y) &= 1 \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

- $\Omega = [0,1]^2 \subset \mathbb{R}^2$ we consider the PDE

$$\begin{aligned} -\Delta u(x,y) &= 20\exp(-20((x-0.5)^2 + (y-0.5)^2)) \quad \text{on } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

In `python` file provided at the web page you find a `FEniCS` implementation of the residual based error estimator from the lecture. The residual is given as a DG0-function ("Discontinuos Galerkin of degree 0", i.e. piecewise constant with respect to the triangulation) named `cell_residual` which equals the residual of the respective cell at each cell of the triangulation.

a) Plot the DG0-estimator function for different mesh sizes. Which cells are those with the largest errors? Can you explain this?

b) Add a routine that computes the overall error estimate of the respective solution. Plot the overall error estimate with respect to the mesh size.

Of course, the next step would be the implementation of the full adaptive finite element algorithm (solve ⟶ estimate ⟶ mark ⟶ refine). We leave this for further lectures on this topic...

Submission til January 23 directly to your tutor.