Scientific Computing I
(Wissenschaftliches Rechnen I)

Winter term 2019/20

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2^{nd} excercise sheet

Submission on October 24, before the lecture

Excercise 1. (2 + 2 = 4 points)

For $d \in \mathbb{N}$ and $\alpha > 0$ we consider the function

$$f : \mathbb{R}^d \longrightarrow \mathbb{R}, \qquad x \mapsto \|x\|_2^{-\alpha}.$$

a) Compute the range of all α such that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

b) For $p \in (1, \infty)$ determine all α such that $f \in W^{1,p}([-1,1]^d)$.

Excercise 2. (2 + 2 = 4 points)

We consider the absolute value function f and the so called Heaviside function g defined as follows:

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto |x|,$$
$$g : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

a) Compute the weak derivative of f. Use directly the definition and not Excercise 3a.

b) Show that g is not weakly differentiable.

Which kind of object could be a candidate for an even "weaker" kind of derivative[1] of g?

[1]This kind of derivative, the *distributional derivative*, will not be required in this lecture...

Excercise 3. (3 + 2 = 5 points)

a) Let $\{\Omega_j\}_{j=1,\dots,J}$ be a partition of a domain Ω into piecewise smooth subdomains Ω_j, i.e. $\Omega \subset \cup_{j=1}^{J} \overline{\Omega_j}$ and $\Omega_j \cap \Omega_{j'} = \emptyset$ for $j \neq j'$. Show that every $u \in C^0(\overline{\Omega})$ such that $u|_{\Omega_j} \in C^1(\overline{\Omega}_j)$ for all $j = 1, \dots, J$ is weakly differentiable on Ω.

b) Let $\Omega \subset \mathbb{R}^n$ be a domain and $B \in \mathbb{R}^{n\times n}$ invertible. With $B(\Omega)$ we denote the image of Ω under B. With a function $u \in C^1(B(\Omega))$ we associate $v := u \circ B \in C^1(\Omega)$.

Express the $W^{1,p}$-norm on $B(\Omega)$, i.e. $\|u\|^p_{W^{1,p}(\Omega)} := \int_{B(\Omega)} |u|^p \, dx + \int_{B(\Omega)} |\nabla u|_2^p \, dx$, in terms of v and integrals over Ω.

Excercise 4. (3 + 4 = 7 points)

a) Let $u, v \in W^{1,2}(\Omega)$ for some domain $\Omega \subset \mathbb{R}^n$. Show that $uv \in W^{1,1}(\Omega)$.

b) Let Ω be a bounded domain and $f \in C^1(\mathbb{R})$ such that $|f'| \leq M$ with some constant $M > 0$. Show that it holds:

$$u \in W^{1,p}(\Omega) \quad \Longrightarrow \quad f \circ u \in W^{1,p}(\Omega)$$

for $p \in [1, \infty)$.

Hint: Determine a “intuitive” candidate for the respective weak derivatives. Use the density of $C^\infty(\Omega)$ in $W^{k,p}(\Omega)$ for $p \in [1, \infty)$, $k \in \mathbb{N}$.